

TMUA Paper 2 Masterclass

Part 1: Foundations of Mathematical Logic
Statements, Truth, Negation, AND, OR and Logical Equivalence

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Welcome to TMUA Paper 2

TMUA Paper 1 asks you to apply mathematical knowledge.
Paper 2 places much greater emphasis on

mathematical reasoning.

You must be able to understand how mathematical statements relate to one another and how valid mathematical arguments are constructed.

Key Idea

Paper 2 is not simply another collection of algebra, trigonometry and calculus questions.
It asks whether you can **think logically about mathematics**.

What Does Paper 2 Test?

Paper 2 develops three broad areas:

Logic of Arguments

Mathematical Proof

Identifying Errors in Proofs

The mathematics from Paper 1 still matters.
But now another question becomes central:

Does the conclusion actually follow?

The Central Question of Mathematical Logic

Suppose we know some mathematical facts.
What are we genuinely allowed to conclude from them?
That question lies at the heart of mathematical reasoning.
For example,

$$x = 3$$

certainly gives

$$x^2 = 9.$$

But from

$$x^2 = 9$$

can we conclude

$$x = 3?$$

The Central Question of Mathematical Logic (continued)

No.
Because

$$x = -3$$

is also possible.

Correct Algebra Is Not Enough

Consider the argument:

$$x^2 = 9$$

$$\therefore x = 3.$$

The calculation may look familiar.
But the deduction is incomplete.
The correct conclusion is

$$x = \pm 3.$$

TMUA Thinking

Paper 2 repeatedly asks you to distinguish between something that *looks plausible* and something that is *logically justified*.

Mathematics Has a Precise Language

Words such as

and

or

not

and later

if

only if

if and only if

have precise meanings in mathematics.
Everyday language can tolerate ambiguity.
Mathematical arguments cannot.

Part 1 Roadmap

In this lesson we develop the foundations:

- 1 What is a mathematical statement?
- 2 True and false
- 3 Truth values
- 4 Expressions involving variables
- 5 Compound statements
- 6 Negation: *not*
- 7 Combining statements using *and*
- 8 Mathematical *or*
- 9 Logical equivalence
- 10 Negating compound statements
- 11 De Morgan's laws
- 12 TMUA-style reasoning

What Is a Statement?

For the purposes of TMUA logic, we can think of a mathematical statement as a sentence that is definitely

true

or definitely

false.

It cannot be both.

For example,

24 is divisible by 2

is a statement.

It is true.

A Statement Does Not Have to Be True

Consider

17 is divisible by 4.

This is false.

But it is still a statement.

Why?

Because it has a definite truth value.

Truth value = False

Key Idea

“Statement” does not mean “true statement.”

A false statement is still a statement.

Examples of Statements

Each of the following is a statement:

$$2 + 3 = 5$$

$$7^2 = 48$$

$\sqrt{2}$ is irrational

The sum of two odd integers is even.

Some are true.

Some are false.

Some may require significant mathematical work before we can determine their truth.

Truth Is Different From Knowing the Truth

Consider a very complicated mathematical claim.

Perhaps we cannot immediately decide whether it is true.

That does not automatically prevent it from being a statement.

The important distinction is:

having a truth value

versus

knowing the truth value.

These are not the same thing.

Some Sentences Are Not Statements

Consider:

What is your name?

Is this true?

Is it false?

Neither description makes sense.

Similarly,

Close the door!

is a command, not something to which we assign a truth value.

Therefore these are not statements in the logical sense being used here.

TMUA Does Not Need Philosophical Complications

There are deep philosophical questions about:

- truth;
- vagueness;
- paradoxes;
- different systems of logic.

These can be fascinating.

But they are not the purpose of TMUA Paper 2.

For this course we use the practical framework needed for mathematical reasoning in the test.

Is This a Statement?

Consider

$$x > 5.$$

Is it true?

If

$$x = 8,$$

yes.

If

$$x = 2,$$

no.

So without additional information about x ,

$$\boxed{x > 5}$$

does not yet have a fixed truth value.

Open Expressions

An expression such as

$$x^2 = 9$$

depends on the value of x .

For example,

$$x = 3 \Rightarrow x^2 = 9$$

is true,

whereas

$$x = 4 \Rightarrow x^2 = 9$$

is false.

Later, quantifiers such as

$$\boxed{\text{for all}}$$

Open Expressions (continued)

and

there exists

will turn expressions of this kind into complete statements.

A Compound Statement Can Still Be Definite

Consider:

If an integer x is divisible by 4, then x is divisible by 2.

Although x is not a particular number, the entire mathematical claim has a definite truth value. It is true.

This illustrates why we must pay attention to the **whole logical structure** of a sentence.

Truth Values

The **truth value** of a statement simply tells us whether the statement is

T

or

F .

For example,

A : 12 is even

has truth value

T .

Whereas

B : 12 is prime

Truth Values (continued)

has truth value

F .

Representing Statements by Letters

Instead of repeatedly writing long sentences, we can use letters.
For example,

A : 21 is divisible by 3

and

B : 21 is divisible by 6.

Then we can study combinations such as

A and B ,

A or B ,

and

not A .

This lets us study the logical structure independently of the particular mathematics.

Negation: The Word “Not”

Suppose

A : 29 is prime.

The negation of A is

not A : 29 is not prime.

Exactly one of

A

and

not A

is true.

Truth Table for Negation

The behaviour of negation can be represented as

A	not A
T	F
F	T

So:

$A \text{ true} \Rightarrow \text{not } A \text{ false}$

and

$A \text{ false} \Rightarrow \text{not } A \text{ true.}$

Important TMUA Note About Truth Tables

Truth tables are extremely useful for learning logic.

They help us understand why logical rules work.

However, TMUA candidates are not required to use formal truth-table notation in the examination.

TMUA Thinking

Use truth tables as a **learning and checking tool**.

The deeper objective is to understand the reasoning well enough that the logical relationships become intuitive.

Double Negation

Suppose A is true.

Then

not A

is false.

Therefore

not not A

is true again.

Hence

not not A

has the same truth value as

A .

Repeated Negations

Consider

not not not not A .

Every “not” reverses the truth value.

Therefore:

even number of negations $\Rightarrow$ same truth value as A

and

odd number of negations $\Rightarrow$ opposite truth value.

The Scope of “Not” Matters

Consider

$\text{not } A \text{ or } B.$

This should be interpreted as

$(\text{not } A) \text{ or } B,$

not automatically as

$\text{not}(A \text{ or } B).$

These are different statements.

Common Mistake

Logical brackets matter.

Changing the scope of a negation can completely change the meaning of a statement.

Combining Statements Using “And”

Suppose

$A :$ 12 is even

and

$B :$ 12 is divisible by 3.

Then

$A \text{ and } B$

means:

12 is even and 12 is divisible by 3.

Both components are true.

Therefore the compound statement is true.

When Is A and B True?

For

A and B

to be true,

both statements must be true.

The truth table is

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

Only one row produces a true compound statement.

Example

Let

$A : 15 > 10$

and

$B : 15$ is even.

Then

$A = T$

but

$B = F.$

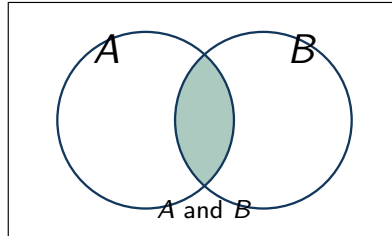
Therefore

A and $B = F.$

One false component is enough to make an “and” statement false.

A Diagrammatic Interpretation of AND

Think of a region where A is true and another where B is true.



The overlap represents situations in which

A and B

are both true.

The Word “Or” Is Important

Everyday English often uses “or” to mean

one or the other, but not both.

For example:

“Would you like tea or coffee?”

may suggest choosing exactly one.

Mathematical logic uses “or” differently.

Mathematical OR Is Inclusive

In TMUA mathematical logic,

A or B

means

A is true, or B is true, or both are true.

This is called

inclusive OR.

So A and B being simultaneously true does **not** make A or B false.

Truth Table for OR

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

Thus

A or B

is false only when

A and B are both false.

Example

Let

A : 10 is even

and

B : $10 > 7$.

Both are true.

Therefore

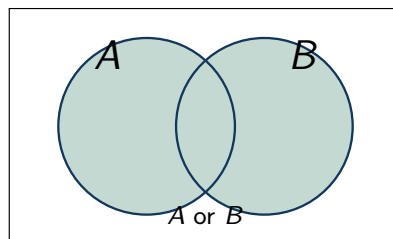
A or B

is also true.

Common Mistake

Do not reject “or” simply because both statements happen to be true.
In mathematical logic, “or” is inclusive unless explicitly stated otherwise.

Diagrammatic Interpretation of OR



Everywhere inside either circle is included.
That includes the overlap.

AND vs OR

Compare:

A and B

with

A or B .

For AND:

both must be true.

For OR:

at least one must be true.

This distinction sounds simple,
but many more complicated logical questions reduce to it.

Building More Complicated Statements

We can combine more than two statements.
For example,

A and $(B$ or $C)$.

To evaluate this, work from the inside:

B or C

first,
then combine the result with A .
This is similar to the role of brackets in ordinary algebra.

Example

Suppose

$$A = T, \quad B = F, \quad C = T.$$

Evaluate

$$A \text{ and } (B \text{ or } C).$$

First,

$$B \text{ or } C = F \text{ or } T = T.$$

Therefore

$$A \text{ and } (B \text{ or } C) = T \text{ and } T = \boxed{T}.$$

Brackets Can Matter

Compare

$$A \text{ or } (B \text{ and } C)$$

with

$$(A \text{ or } B) \text{ and } C.$$

These are not automatically equivalent.
As in algebra,

$$a + (bc)$$

is not generally the same as

$$(a + b)c.$$

Logical structure must be read carefully.

What Does Logically Equivalent Mean?

Two statements are **logically equivalent** when they have the same truth value under the same circumstances.

Symbolically, we might informally think:

$$\text{Statement 1} \iff \text{Statement 2}$$

because whenever one is true, so is the other,
and whenever one is false, so is the other.

A Simple Example

Consider

A or B

and

B or A .

Their truth does not depend on the order.

Hence

$$A \text{ or } B \quad \text{and} \quad B \text{ or } A$$

are logically equivalent.

Similarly,

$$A \text{ and } B \quad \text{and} \quad B \text{ and } A$$

are logically equivalent.

Associativity of AND

Compare

A and $(B \text{ and } C)$

with

$(A \text{ and } B) \text{ and } C.$

Both are true exactly when

A, B, C

are all true.

Therefore they are logically equivalent.

So we can write unambiguously

$A \text{ and } B \text{ and } C.$

Associativity of OR

Similarly,

$A \text{ or } (B \text{ or } C)$

and

$(A \text{ or } B) \text{ or } C$

have the same truth values.

Both mean:

$\text{at least one of } A, B, C \text{ is true.}$

Hence we can write

$A \text{ or } B \text{ or } C$

without ambiguity.

A More Subtle Negation

Suppose the statement is

A and B .

What is its negation?

A common mistake is to write

$(\text{not } A)$ and $(\text{not } B)$.

But that is too strong.

To make

A and B

false, we need only one component to fail.

Negating AND

The negation of

A and B

is

$(\text{not } A) \text{ or } (\text{not } B)$.

Why?

Because A and B fails whenever:

- A is false; or
- B is false; or
- both are false.

Thus

$\text{not}(A \text{ and } B) \equiv (\text{not } A) \text{ or } (\text{not } B)$.

Example: Negating AND

Consider

x is even and x is prime.

Its negation is

x is not even or x is not prime.

For integer x , we could also express “not even” as “odd”, giving

x is odd or x is not prime.

Negating OR

Now consider

A or B .

For this statement to be false,
both A and B must be false.
Therefore

$\text{not}(A \text{ or } B) \equiv (\text{not } A) \text{ and } (\text{not } B).$

De Morgan's Laws

The two important relationships are

$$\text{not}(A \text{ and } B) \equiv (\text{not } A) \text{ or } (\text{not } B)$$

and

$$\text{not}(A \text{ or } B) \equiv (\text{not } A) \text{ and } (\text{not } B).$$

These are commonly known as

De Morgan's Laws.

A Useful Memory Pattern

When a negation passes through a compound statement:

AND changes to OR

and

OR changes to AND

while each component is negated.

So:

$$\text{not}(A \text{ and } B)$$

becomes

$$(\text{not } A) \text{ or } (\text{not } B).$$

Do Not Memorise De Morgan Mechanically

Understanding is safer than memorisation.

Suppose someone claims:

“Ali is tall and Ali is left-handed.”

For this whole statement to be false,
it is enough that Ali is not tall,

or

that Ali is not left-handed.

We do not need both failures.

That is why AND becomes OR under negation.

TMUA Example 1

Suppose

$$A = T, \quad B = F.$$

Which statement is true?

- (A) A and B
- (B) (not A) and B
- (C) A or B
- (D) not(A or B)
- (E) (not A) or B

TMUA Example 1: Solution

Given

$$A = T, \quad B = F,$$

we have

$$A \text{ and } B = F,$$

$$(\text{not } A) \text{ and } B = F \text{ and } F = F,$$

while

$$A \text{ or } B = T \text{ or } F = T.$$

Therefore

$$\boxed{(C)}.$$

TMUA Example 2

Let

$$A : \quad n \text{ is even}$$

and

$$B : \quad n \text{ is divisible by 3.}$$

What is the negation of

$$A \text{ and } B?$$

TMUA Example 3

Suppose the statement

$A \text{ or } B$

is false.

What can we conclude?

Since OR is false only when both components are false,

$$\boxed{A = F}$$

and

$$\boxed{B = F}.$$

This is stronger information than merely knowing one component.

TMUA Example 4

Suppose

$A \text{ and } B$

is false.

Can we conclude that A is false?

No.

Possibilities include

$$(T, F), \quad (F, T), \quad (F, F).$$

Therefore the only justified conclusion is:

$$\boxed{\text{at least one of } A, B \text{ is false.}}$$

A Classic Logical Trap

From

A and B

being false,
do **not** conclude

A and B are both false.

The correct statement is

A is false or B is false.

Remember that mathematical OR is inclusive.

So both may indeed be false,
but that is not required.

Logic and Sets

There is a useful analogy between logic and sets.

Logic	Sets
A and B	$A \cap B$
A or B	$A \cup B$
not A	A^c

This gives another way to visualise logical statements.

TMUA Thinking

If you understand Venn diagrams from probability or set theory, use that intuition to strengthen your understanding of logic.

De Morgan Through Sets

The logical identity

$$\text{not}(A \text{ and } B)$$

corresponds to

$$(A \cap B)^c.$$

De Morgan's law for sets gives

$$(A \cap B)^c = A^c \cup B^c.$$

Likewise,

$$(A \cup B)^c = A^c \cap B^c.$$

The same structure appears in both logic and set theory.

Logic Is Not Separate From Mathematics

Consider:

$$x^2 > 4.$$

For real x ,

$$x^2 > 4$$

means

$$\boxed{x > 2 \text{ or } x < -2}.$$

Notice the logical word:

$$\boxed{\text{or}}.$$

Algebraic solution sets already contain logical structure.

Another Example

Suppose

$$(x - 2)(x + 5) = 0.$$

Then

$$x - 2 = 0$$

or

$$x + 5 = 0.$$

Therefore

$$x = 2 \text{ or } x = -5.$$

The algebraic reasoning relies on a logical alternative.
Mathematical logic is embedded throughout ordinary mathematics.

Logic Inside Geometry

Suppose a point lies

$$\text{inside circle } A \text{ and inside circle } B.$$

That means it lies in the intersection.

If instead it lies

$$\text{inside } A \text{ or inside } B,$$

it lies in their union.

This is why diagrammatic reasoning can make logical structure much easier to see.

Logic Inside Probability

Suppose A and B are events.

Then

$$A \cap B$$

means both occur,
while

$$A \cup B$$

means at least one occurs.

Similarly,

$$A^c$$

means A does not occur.

Thus logic, sets and probability share a common structural language.

Challenge 1

Suppose

$$A = F, \quad B = T, \quad C = F.$$

Determine the truth value of

$$(\text{not } A) \text{ and } (B \text{ or } C).$$

Challenge 2

Negate:

$$x > 0 \text{ and } x < 10.$$

Challenge 3

Negate:

$$x \leq 2 \text{ or } x \geq 7.$$

Challenge 4

Suppose

$$A \text{ or } B = T$$

and

$$A = F.$$

What must be true?

Since at least one of A, B must be true,
and A is false,
we must have

$$\boxed{B = T}.$$

This is a simple but important deductive pattern.

Challenge 5

Suppose

$$A \text{ and } B = T.$$

What can you conclude?

Immediately,

$$\boxed{A = T}$$

and

$$\boxed{B = T}.$$

There is no alternative possibility.

Compare this with knowing that A and B is false, which gives much less information.

Common Error 1: Treating OR as Exclusive

Wrong interpretation:

A or B

means exactly one is true.

Correct TMUA interpretation:

A or $B =$ at least one is true.

Both may be true.

Common Error 2: Negating AND Incorrectly

Wrong:

$$\text{not}(A \text{ and } B) = (\text{not } A) \text{ and } (\text{not } B).$$

Correct:

$$\text{not}(A \text{ and } B) = (\text{not } A) \text{ or } (\text{not } B).$$

Always ask:

What would make the original statement fail?

Common Error 3: Ignoring Boundaries

The negation of

$$x > 5$$

is not

$$x < 5.$$

It is

$$\boxed{x \leq 5}.$$

Similarly,

$$\text{not}(x \geq 5)$$

is

$$\boxed{x < 5}.$$

Negation must cover every case excluded by the original statement.

Common Error 4: Ignoring Brackets

These are different:

$$(\text{not } A) \text{ or } B$$

and

$$\text{not}(A \text{ or } B).$$

The second is equivalent to

$$(\text{not } A) \text{ and } (\text{not } B).$$

Read the logical structure before evaluating truth values.

Part 1 Concept Map

Statement

has a

Truth Value

which is

T or F .

Statements can then be transformed or combined using

NOT

AND

OR.

From these simple ingredients we can construct increasingly complex mathematical arguments.

The Three Fundamental Operations

not A

reverses the truth value.

A and B

is true only when both are true.

A or B

is true when at least one is true.

These rules are the foundations on which the next stages of Paper 2 logic are built.

De Morgan's Laws

Remember:

$$\text{not}(A \text{ and } B) \equiv (\text{not } A) \text{ or } (\text{not } B)$$

and

$$\text{not}(A \text{ or } B) \equiv (\text{not } A) \text{ and } (\text{not } B).$$

But do not merely memorise them.

Understand what would make the original compound statement false.

Self-Check

Before moving to Part 2, make sure you can answer:

- ❶ What is a statement?
- ❷ What is a truth value?
- ❸ What is the negation of a statement?
- ❹ When is A and B true?
- ❺ When is A or B false?
- ❻ Why is mathematical OR inclusive?
- ❼ What does logically equivalent mean?
- ❽ How do we negate A and B ?
- ❾ How do we negate A or B ?
- ❿ Why do brackets matter?

The Mathematical Habit to Develop

Whenever you read a mathematical sentence, ask:

What exactly is being claimed?

Then:

When is it true?

Then:

When is it false?

And finally:

What am I logically allowed to conclude?

That habit is central to TMUA Paper 2.

Coming in Part 2

Implication and Conditional Statements

If A then B

A if B

A only if B

A if and only if B

We will learn why these apparently similar phrases can encode very different logical information.

**Think Precisely.
Reason Carefully.
Conclude Only What Follows.**

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TMUA Paper 2 Masterclass

Part 2: Implication and Conditional Statements If, Only If, and If and Only If

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From Statements to Arguments

In Part 1 we studied:

statements

truth values

and the logical operations

not

and

or.

We now move to one of the most important structures in mathematics:

If A , then B .

This is the language of mathematical implication.

Why “If” Matters

Mathematics constantly uses statements such as

If $x = 3$, then $x^2 = 9$,

If n is divisible by 4, then n is even,

and

If $f'(x) > 0$, then f is increasing

under the relevant assumptions.

These statements describe how one mathematical fact **forces** another.

The Central Question

Whenever you see

If A , then B ,

ask:

What exactly does this allow me to conclude?

and equally importantly,

What does it NOT allow me to conclude?

Many errors in mathematical reasoning come from confusing these two questions.

Part 2 Roadmap

We will study:

- ① Statements of the form “if A , then B ”
- ② Antecedent and consequent
- ③ When an implication is true or false
- ④ Why a false hypothesis causes difficulty
- ⑤ Equivalent form: not A or B
- ⑥ A if B
- ⑦ A only if B
- ⑧ A if and only if B
- ⑨ Reading mathematical conditions correctly
- ⑩ TMUA-style reasoning problems

The Basic Conditional Statement

Consider

If A , then B .

Here:

A

is the condition or hypothesis,
and

B

is the conclusion.
The statement says:

whenever A is true, B must also be true.

Antecedent and Consequent

In

if A then B ,

A is often called the

antecedent

and B the

consequent.

You do not need complicated terminology to reason correctly.
The essential structure is simply

$$A \longrightarrow B.$$

Whenever A happens, B must happen.

A Mathematical Example

Consider

$$\text{If } x = 3, \text{ then } x^2 = 9.$$

If the first statement is true,

$$x = 3,$$

then

$$x^2 = 9$$

must be true.
So the implication works.

Another Example

Consider

If an integer n is divisible by 4, then n is even.

Why?

If n is divisible by 4,

$$n = 4k$$

for some integer k .

Then

$$n = 2(2k),$$

so n is even.

Therefore the implication is true.

The Crucial Case

Consider

if A then B .

Suppose

$$A = T$$

but

$$B = F.$$

Then the implication has failed.

Why?

Because A occurred,

but the promised consequence B did not.

Therefore

$$T \rightarrow F$$

The Crucial Case (continued)

is false.

Truth Table for Implication

The truth structure is

A	B	if A then B
T	T	T
T	F	F
F	T	T
F	F	T

The key fact is:

if A then B is false only when $A = T$, $B = F$.
--

The First Two Rows Are Natural

If

$$A = T, \quad B = T,$$

then

$$A \rightarrow B$$

is true.

If

$$A = T, \quad B = F,$$

then the implication is false.

These cases usually feel intuitive.

The interesting cases occur when

$$\boxed{A = F}.$$

What Does the Statement Say When A Is False?

Suppose

$$\boxed{\text{If } A, \text{ then } B}$$

is true.

It tells us what must happen when A is true.

But if

$$A = F,$$

the statement places no requirement on B .

Thus B may be true,

or B may be false.

Example

Consider the true statement:

If an integer is divisible by 4, then it is even.

Now consider the integer

6.

Six is not divisible by 4,
but it is even.

Does this contradict the implication?

No.

The statement never claimed that every even integer must be divisible by 4.

Another Example

Consider

If $x = 2$, then $x^2 = 4$.

Now choose

$$x = -2.$$

The hypothesis

$$x = 2$$

is false,
while

$$x^2 = 4$$

is true.

This does not make the implication false.

A Strange-Looking Example

Consider

$$\boxed{\text{If } 0 = 1, \text{ then } 2 + 2 = 5.}$$

The hypothesis

$$0 = 1$$

is false.

Therefore we never encounter the situation in which the hypothesis is true but the conclusion is false. Under mathematical implication, the whole conditional statement is therefore true.

Why This Can Feel Strange

Everyday language often suggests a causal relationship.
For example,

“If it rains, I will use an umbrella.”

may carry additional conversational implications.
Formal mathematical logic strips those away.
It asks only:

$\boxed{\text{Is there a case where } A \text{ is true but } B \text{ is false?}}$

If yes, the implication is false.
If no, it is true.

A Powerful Interpretation

Instead of memorising four rows, think:

$$A \rightarrow B$$

forbids exactly one situation:

$$A \text{ true and } B \text{ false.}$$

Everything else is compatible with the implication.

This viewpoint becomes extremely useful when we study converse and contrapositive statements.

Can We Rewrite an Implication?

Recall:

$$A \rightarrow B$$

is false exactly when

$$A = T, \quad B = F.$$

Now consider

$$(\text{not } A) \text{ or } B.$$

When is this false?

An OR statement is false only when both parts are false.

So we require

$$\text{not } A = F$$

and

Can We Rewrite an Implication? (continued)

$$B = F.$$

Hence

$$A = T, \quad B = F.$$

Exactly the same failure case.

Important Logical Equivalence

Therefore

$$\boxed{\text{if } A \text{ then } B \equiv (\text{not } A) \text{ or } B.}$$

This is one of the most useful ways to understand implication.
It says:

$$\boxed{A \rightarrow B}$$

is equivalent to saying

$$\boxed{A \text{ is false, or } B \text{ is true.}}$$

Remember that OR is inclusive.

Checking With a Truth Table

A	B	not A	(not A) or B	$A \rightarrow B$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The final two columns are identical.
Therefore the statements are logically equivalent.

Another Equivalent Description

Since

$$A \rightarrow B$$

fails exactly when

$$A \text{ and not } B$$

is true,
we can also write

$$A \rightarrow B \equiv \text{not}(A \text{ and not } B).$$

Using De Morgan's law,

$$\text{not}(A \text{ and not } B) \equiv (\text{not } A) \text{ or } B.$$

The structures agree.

Using “If” in a Different Order

Compare:

If A , then B

and

B if A .

These mean the same thing.
Thus

$B \text{ if } A \equiv \text{if } A \text{ then } B.$

The word order changes,
but the logical direction does not.

Example

Consider:

$x^2 = 9$ if $x = 3$.

The phrase after “if” gives the condition.
So this means

if $x = 3$, then $x^2 = 9$.

A useful reading habit is:

$B \text{ if } A \longrightarrow A \rightarrow B.$

TMUA Translation Habit

Whenever you encounter

B if A ,

mentally rewrite it as

$\boxed{\text{if } A \text{ then } B.}$

For example,

n is even if n is divisible by 4

becomes

$\boxed{\text{if } n \text{ is divisible by 4, then } n \text{ is even.}}$

This removes linguistic ambiguity.

Now Comes the Dangerous Phrase

Consider

$\boxed{A \text{ only if } B.}$

Students frequently reverse this statement.

The correct meaning is

$\boxed{\text{if } A \text{ then } B.}$

Therefore

$\boxed{A \text{ only if } B \equiv A \rightarrow B.}$

How to Understand “Only If”

Suppose

A only if B .

The phrase means that A cannot happen unless B is present.

Therefore, if A occurs,

B must occur.

Hence

$$A \rightarrow B.$$

Think:

B is required for A .

Example

Consider

$$x = 3 \text{ only if } x^2 = 9.$$

This means

$$\text{if } x = 3, \text{ then } x^2 = 9.$$

It does **not** mean

$$x^2 = 9 \Rightarrow x = 3.$$

Why not?

Because

$$x = -3$$

also satisfies $x^2 = 9$.

IF vs ONLY IF

These two phrases are easy to confuse:

$$A \text{ if } B$$

means

$$B \rightarrow A,$$

whereas

$$A \text{ only if } B$$

means

$$A \rightarrow B.$$

The arrows point in opposite directions.

The Translation Table

Words	Logical direction
If A then B	$A \rightarrow B$
B if A	$A \rightarrow B$
A only if B	$A \rightarrow B$
A if B	$B \rightarrow A$
B only if A	$B \rightarrow A$

TMUA Thinking

When the language feels confusing, rewrite everything in “if ... then ...” form.

Two Directions at Once

Now consider

$$A \text{ if and only if } B.$$

This combines two implications:

$$A \text{ if } B$$

and

$$A \text{ only if } B.$$

Therefore it means

$$(B \rightarrow A) \text{ and } (A \rightarrow B).$$

The Meaning of If and Only If

Thus

$$A \text{ if and only if } B$$

means both

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

So each statement forces the other.

We can picture this as

$$A \iff B.$$

They are true in exactly the same circumstances.

Truth Structure of “If and Only If”

A	B	A if and only if B
T	T	T
T	F	F
F	T	F
F	F	T

The statement is true when

A and B have the same truth value.

A Simple Mathematical Example

For an integer n ,

n is even if and only if n is not odd.

If n is even,
then it is not odd.
And if n is not odd,
then, for an integer, it is even.
Both directions hold.

Another Example

For a real number x ,

$$x^2 = 0 \text{ if and only if } x = 0.$$

Direction 1:

$$x^2 = 0 \Rightarrow x = 0.$$

Direction 2:

$$x = 0 \Rightarrow x^2 = 0.$$

Both are valid.

Therefore the biconditional is true.

When “If and Only If” Fails

Consider the claim:

$$x^2 = 9 \text{ if and only if } x = 3.$$

One direction works:

$$x = 3 \Rightarrow x^2 = 9.$$

But the reverse direction fails:

$$x^2 = 9 \not\Rightarrow x = 3$$

because $x = -3$ is possible.

Therefore the entire “if and only if” statement is false.

Conditions Matter

Consider

$$A \rightarrow B.$$

This tells us that B is required whenever A holds.
Later we will formalise this language using

necessary

and

sufficient.

For now, develop the more fundamental habit:

Which statement forces which?

Draw the Arrow

Whenever language becomes confusing,
write an arrow.
For example:

"A square is a rectangle."

means

square $\rightarrow$ rectangle.

But

rectangle $\rightarrow$ square

is false.
A rectangle need not have all four sides equal.

Example: Divisibility

For integers,

$$12 \mid n$$

implies

$$3 \mid n.$$

So

$$12 \mid n \rightarrow 3 \mid n.$$

But

$$3 \mid n$$

does not imply

$$12 \mid n.$$

Example: Divisibility (continued)

For example,

$$n = 6.$$

Example: Geometry

If a quadrilateral is a square,
then it is a rectangle.
Thus

$$\text{square} \rightarrow \text{rectangle}.$$

If a quadrilateral is a rectangle,
can we conclude that it is a square?
No.
Therefore the reverse implication does not follow.

What Can We Deduce?

Suppose we know

$$A \rightarrow B$$

and we also know

$$A = T.$$

Then necessarily

$$B = T.$$

This is the basic forward use of an implication.

Example

Suppose:

If n is divisible by 6, then n is divisible by 3.

We are told that

n is divisible by 6.

Therefore

n is divisible by 3.

This is a valid deduction.

What If We Know B ?

Suppose

$$A \rightarrow B$$

and

$$B = T.$$

Can we conclude

$$A = T?$$

No.

The implication does not generally work backwards.

This is one of the most important logical warnings in Paper 2.

Example

We know

divisible by 6 $\rightarrow$ divisible by 3.

Suppose n is divisible by 3.

Can we conclude that n is divisible by 6?

No.

For example,

$$n = 9.$$

So knowing the conclusion is true does not establish the hypothesis.

What If We Know A Is False?

Suppose

$$A \rightarrow B$$

and

$$A = F.$$

Can we determine B ?

No.

B may be true or false.

Therefore:

$A \text{ false} \not\Rightarrow B \text{ false.}$
--

This is another major reasoning trap.

Four Pieces of Information

Given

$$A \rightarrow B,$$

consider what happens if we additionally know:

Known	What follows?
A true	B true
A false	nothing definite about B
B true	nothing definite about A
B false	A false

The final row will become extremely important in Part 3.

TMUA Example 1

Suppose the statement

If A then B

is true.

Which situation is impossible?

- (A) $A = T, B = T$
- (B) $A = T, B = F$
- (C) $A = F, B = T$
- (D) $A = F, B = F$

TMUA Example 1 (continued)

$$\boxed{(B)}.$$

TMUA Example 2

Suppose

$$A \rightarrow B$$

is true and

$$B = F.$$

What can we conclude?

If A were true,
then B would have to be true.

But B is false.

Therefore

$$\boxed{A = F}.$$

This reasoning anticipates the contrapositive.

TMUA Example 3

Which statement means the same as

P only if Q ?

(A) $P \rightarrow Q$

(B) $Q \rightarrow P$

(C) P or Q

(D) P and Q

The correct answer is

(A).

TMUA Example 4

Which statement means the same as

P if Q ?

Rewrite:

P if Q

as

if Q then P .

Therefore

$Q \rightarrow P$.

Notice how easily “if” and “only if” can reverse the direction.

TMUA Example 5

Suppose

P if and only if Q .

We know P is false.

What follows?

Since

$$P \leftrightarrow Q$$

requires the two statements to have the same truth value,

Q must also be false.

Translation Problem 1

Translate:

n is divisible by 8 only if n is even.

The phrase has the form

A only if B .

Therefore

n divisible by 8 $\rightarrow$ n even.

This implication is true.

Translation Problem 2

Translate:

$$x^2 = 4 \text{ if } x = 2.$$

This has the form

$$A \text{ if } B.$$

Therefore

$$B \rightarrow A.$$

So:

$$x = 2 \rightarrow x^2 = 4.$$

Translation Problem 3

Translate:

$$x = 0 \text{ if and only if } x^2 = 0.$$

This means both

$$x^2 = 0 \rightarrow x = 0$$

and

$$x = 0 \rightarrow x^2 = 0.$$

Both are true for real x .

Translation Problem 4

Consider:

n is divisible by 6 if n is divisible by 12.

The phrase after “if” is the hypothesis.
Therefore

$$12 \mid n \rightarrow 6 \mid n.$$

Always identify the logical direction before deciding whether the statement is true.

Do Not Read Logic as Ordinary English

In everyday speech,
“if” can suggest:

- causation;
- expectation;
- chronology;
- intention.

Mathematical implication requires none of these.
It describes a relationship between truth values.

Mathematical Maturity

Separate the formal mathematical meaning from the conversational meaning of the sentence.

Implication Is About Information

Suppose

$$A \rightarrow B.$$

Knowing A gives us new information:

$$A \implies B.$$

Knowing B , however, may not tell us whether A occurred.
So implication is directional.

$$A \rightarrow B \neq B \rightarrow A$$

in general.

Think in Terms of Sets

Suppose $A(x)$ and $B(x)$ describe properties.
If

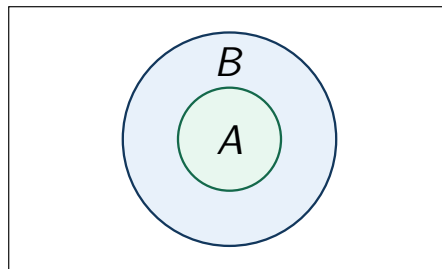
$$A(x) \rightarrow B(x)$$

for every relevant x ,
then every object satisfying A also satisfies B .
Diagrammatically:

$$A \subseteq B.$$

This gives a powerful geometric interpretation of implication.

Set Picture of Implication



If every A -case lies inside the B -region,
then

$$A \rightarrow B.$$

Why the Reverse Need Not Hold

If

$$A \subseteq B,$$

there may be points inside B that are not inside A .

Therefore

$$A \rightarrow B$$

does not automatically give

$$B \rightarrow A.$$

This set interpretation makes the asymmetry of implication visually clear.

Challenge 1

Suppose

$$A \rightarrow B$$

is true.

Which additional information guarantees A is false?

- (A) B is true
- (B) B is false
- (C) A or B is true

Challenge 2

Suppose

$$P \text{ only if } Q$$

is true.

We know Q is true.

Must P be true?

Challenge 3

Suppose

P if Q

is true,
and Q is true.
What follows?
Since

P if Q

means

$Q \rightarrow P,$

we conclude

P is true.

Challenge 4

Suppose

A if and only if B

is true.
Which pairs of truth values are possible?

(T, T)

and

$(F, F).$

The mixed cases

$(T, F), \quad (F, T)$

are impossible.
Thus A and B always agree in truth value.

Challenge 5

Is the following statement true for every integer n ?

n is even only if n^2 is even.

Translate first:

$$n \text{ even} \rightarrow n^2 \text{ even.}$$

If

$$n = 2k,$$

then

$$n^2 = 4k^2 = 2(2k^2),$$

which is even.

Therefore the statement is true.

Common Error 1: Reversing “If”

Wrong:

$$A \text{ if } B \Rightarrow A \rightarrow B.$$

Correct:

$$A \text{ if } B \iff B \rightarrow A.$$

Read:

“A happens if B happens.”

So B is enough to trigger A .

Common Error 2: Reversing “Only If”

Wrong:

$$A \text{ only if } B \Rightarrow B \rightarrow A.$$

Correct:

$$A \text{ only if } B \iff A \rightarrow B.$$

If A occurs, B is required.

Common Error 3: Assuming the Reverse

From

$$A \rightarrow B,$$

students often conclude

$$B \rightarrow A.$$

This is not generally valid.

Example:

$$x = 3 \rightarrow x^2 = 9$$

but

$$x^2 = 9 \not\Rightarrow x = 3.$$

The reverse statement requires separate justification.

Common Error 4: Rejecting False Hypotheses

Students sometimes think:

$$A = F$$

must make

$$A \rightarrow B$$

false.

That is incorrect.

The implication fails only when

$$A = T, \quad B = F.$$

A false hypothesis does not violate the implication.

Common Error 5: Treating IFF as One Direction

To establish

A if and only if B ,

you need both:

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Showing only one direction is insufficient.

Common Mistake

An “if and only if” statement contains two implications. Never prove only one and stop.

Part 2 Summary

The fundamental implication is

$$A \rightarrow B.$$

It means:

whenever A is true, B must be true.

It is false only in the case

$$A = T, \quad B = F.$$

And it is logically equivalent to

$$(\text{not } A) \text{ or } B.$$

Language Summary

Remember:

$$\text{If } A \text{ then } B \equiv A \rightarrow B$$

$$B \text{ if } A \equiv A \rightarrow B$$

$$A \text{ only if } B \equiv A \rightarrow B$$

and

$$A \text{ if and only if } B \equiv (A \rightarrow B) \text{ and } (B \rightarrow A).$$

A Compact Translation Rule

When you see

$$A \text{ if } B,$$

think

$$B \rightarrow A.$$

When you see

$$A \text{ only if } B,$$

think

$$A \rightarrow B.$$

When you see

$$A \text{ if and only if } B,$$

A Compact Translation Rule (continued)

think

$$A \leftrightarrow B.$$

Self-Check

Before moving to Part 3, make sure you can explain:

- ❶ What does “if A , then B ” mean?
- ❷ When is an implication false?
- ❸ What happens when A is false?
- ❹ Why is $A \rightarrow B$ equivalent to $(\text{not } A) \text{ or } B$?
- ❺ What does A if B mean?
- ❻ What does A only if B mean?
- ❼ What does “if and only if” mean?
- ❽ Why does $A \rightarrow B$ not normally imply $B \rightarrow A$?

The Paper 2 Habit

Do not ask only:

Is this statement true?

Also ask:

What direction does the implication run?

What information is sufficient?

What information is actually necessary?

What can I legitimately conclude?

These distinctions are central to mathematical reasoning.

Coming in Part 3

Converse, Contrapositive and Logical Equivalence

Starting from

$$A \rightarrow B,$$

we will compare it with

$$B \rightarrow A$$

and

$$\text{not } B \rightarrow \text{not } A.$$

Coming in Part 3 (continued)

We will answer one of the most important questions in mathematical logic:

Which of these statements are actually equivalent?

Read the Direction. Respect the Condition. Conclude Only What Follows.

Mathematics Elevate Academy

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TMUA Paper 2 Masterclass

Part 3: Converse, Contrapositive and Logical Equivalence
Understanding What Really Follows

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Where We Are

In Part 2 we studied the implication

$$A \rightarrow B.$$

It means:

whenever A is true, B must be true.

We also learned that the implication is false only when

$$A = T, \quad B = F.$$

Now we ask a deeper question:

What happens if we reverse or negate the statements?

Three Statements to Distinguish

Starting with

$$A \rightarrow B,$$

we will compare:

$$A \rightarrow B$$

with

$$B \rightarrow A$$

and

$$\neg B \rightarrow \neg A.$$

They look similar.

But logically they behave very differently.

Part 3 Roadmap

We will study:

- ① The original implication
- ② The converse
- ③ Why the converse need not be true
- ④ Counterexamples
- ⑤ The contrapositive
- ⑥ Why the contrapositive is equivalent to the original
- ⑦ Valid and invalid deductions
- ⑧ Geometric and set interpretations
- ⑨ Mathematical examples
- ⑩ TMUA-style reasoning

Start With the Original Statement

Suppose we know

$$\boxed{A \rightarrow B}.$$

This says:

$\boxed{\text{If } A \text{ is true, then } B \text{ must be true.}}$

It forbids exactly one situation:

$$\boxed{A = T, \quad B = F}.$$

Keep this structure in mind throughout the lesson.

A Concrete Example

Consider:

If an integer n is divisible by 4, then n is even.

Let

$$A : 4 \mid n$$

and

$$B : n \text{ is even.}$$

Then the statement is

$$A \rightarrow B.$$

Why the Original Statement Is True

If

$$4 \mid n,$$

then

$$n = 4k$$

for some integer k .

But

$$n = 2(2k).$$

Since $2k$ is an integer,

$$2 \mid n.$$

Therefore

Why the Original Statement Is True (continued)

$$4 \mid n \Rightarrow 2 \mid n.$$

What Is the Converse?

Starting from

$$A \rightarrow B,$$

reverse the direction:

$$B \rightarrow A.$$

This reversed implication is called the

converse.

So:

$$\text{Original: } A \rightarrow B$$

$$\text{Converse: } B \rightarrow A.$$

Example of a Converse

Original:

$$4 \mid n \Rightarrow n \text{ is even.}$$

Converse:

$$n \text{ is even} \Rightarrow 4 \mid n.$$

Is the converse true?

No.

Take

$$n = 6.$$

Then 6 is even, but

$$4 \nmid 6.$$

The Central Warning

A true implication

$$A \rightarrow B$$

does **not** automatically imply

$$B \rightarrow A.$$

In symbols:

$$A \rightarrow B \not\Rightarrow B \rightarrow A.$$

Common Mistake

Reversing a true implication is one of the most common logical errors in mathematics.

Example: Squares and Rectangles

Original statement:

If a quadrilateral is a square, then it is a rectangle.

This is true.

The converse is:

If a quadrilateral is a rectangle, then it is a square.

This is false.

A non-square rectangle is a counterexample.

Example: Algebra

Consider

$$x = 3 \Rightarrow x^2 = 9.$$

The converse is

$$x^2 = 9 \Rightarrow x = 3.$$

But

$$x = -3$$

satisfies

$$x^2 = 9$$

without satisfying

$$x = 3.$$

Example: Algebra (continued)

Therefore the converse is false.

How Do We Show an Implication Is False?

Suppose someone claims

$$A \rightarrow B.$$

To disprove it, we need a case where

$$A = T$$

but

$$B = F.$$

Such a case is called a

counterexample.

One valid counterexample is enough to disprove a universal implication.

Counterexample Example

Claim:

$$x^2 > 4 \Rightarrow x > 2$$

for real x .

Try

$$x = -3.$$

Then

$$x^2 = 9 > 4,$$

so the hypothesis is true.

But

$$-3 > 2$$

is false.

Counterexample Example (continued)

Therefore

$$x = -3$$

is a counterexample.

What Makes a Counterexample Valid?

To disprove

$$A \rightarrow B,$$

your example must satisfy

$$\boxed{A}$$

and fail

$$\boxed{B}.$$

An example for which A is false proves nothing.

Why?

Because

$$A \rightarrow B$$

places no requirement on B when A is false.

A Bad Counterexample

Claim:

If n is divisible by 4, then n is even.

Suppose someone chooses

$$n = 3.$$

Then 3 is not even.

But this does not disprove the claim.

Why?

Because the hypothesis is also false:

$$4 \nmid 3.$$

A counterexample must satisfy the hypothesis.

Counterexample Checklist

To test

$$A \rightarrow B,$$

search specifically for

A true and B false.

Do not simply search for an unusual example.

Ask:

Does my example satisfy the hypothesis?

Then:

Does it violate the conclusion?

Only then is it a counterexample.

A Different Transformation

Starting from

$$A \rightarrow B,$$

first reverse the order:

$$B \rightarrow A,$$

and then negate both statements:

$$\neg B \rightarrow \neg A.$$

This is called the

contrapositive.

Original, Converse and Contrapositive

Starting from

$$A \rightarrow B,$$

we have:

Statement	Form
Original	$A \rightarrow B$
Converse	$B \rightarrow A$
Contrapositive	$\neg B \rightarrow \neg A$

Do not confuse

$$B \rightarrow A$$

with

$$\neg B \rightarrow \neg A.$$

Example of the Contrapositive

Original:

If n is divisible by 4, then n is even.

Contrapositive:

If n is not even, then n is not divisible by 4.

For integers, “not even” means “odd.”

Thus:

$$n \text{ odd} \Rightarrow 4 \nmid n.$$

Another Example

Original:

$$x = 3 \Rightarrow x^2 = 9.$$

Contrapositive:

$$x^2 \neq 9 \Rightarrow x \neq 3.$$

This is true.

If

$$x^2 \neq 9,$$

then x certainly cannot equal 3,
because $3^2 = 9$.

The Fundamental Result

The original implication

$$A \rightarrow B$$

and its contrapositive

$$\neg B \rightarrow \neg A$$

are logically equivalent.

That is,

$$A \rightarrow B \equiv \neg B \rightarrow \neg A.$$

This is one of the most important logical equivalences in mathematics.

Truth Table Verification

A	B	$A \rightarrow B$	$\neg B$	$\neg A$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Now evaluate

$$\neg B \rightarrow \neg A.$$

Complete Comparison

A	B	$A \rightarrow B$	$\neg B \rightarrow \neg A$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The final two columns are identical.
Therefore

$$A \rightarrow B \equiv \neg B \rightarrow \neg A.$$

Understanding Without a Truth Table

Recall from Part 2:

$$A \rightarrow B$$

is equivalent to

$$\neg A \text{ or } B.$$

Now consider the contrapositive:

$$\neg B \rightarrow \neg A.$$

Using the same rule,

$$\neg B \rightarrow \neg A$$

is equivalent to

$$\neg(\neg B) \text{ or } \neg A.$$

Understanding Without a Truth Table (continued)

Therefore

$$B \text{ or } \neg A.$$

The Algebra of the Logic

We have

$$A \rightarrow B \equiv \neg A \vee B.$$

And

$$\neg B \rightarrow \neg A \equiv B \vee \neg A.$$

But OR is commutative:

$$B \vee \neg A \equiv \neg A \vee B.$$

Therefore

$$A \rightarrow B \equiv \neg B \rightarrow \neg A.$$

What Does the Contrapositive Say?

Suppose

$$A \rightarrow B.$$

If A always forces B ,
then whenever B has failed,
 A cannot have occurred.
So:

$$A \rightarrow B$$

naturally gives

$$\neg B \rightarrow \neg A.$$

This is not a trick.

It is the same logical information viewed from the opposite direction.

Example: Divisibility

Suppose

$$6 \mid n \Rightarrow 3 \mid n.$$

Then automatically,

$$3 \nmid n \Rightarrow 6 \nmid n.$$

Why?

If n were divisible by 6,

it would necessarily be divisible by 3.

Therefore failure of divisibility by 3 rules out divisibility by 6.

Example: Geometry

Original:

square $\rightarrow$ rectangle.

Contrapositive:

not a rectangle $\rightarrow$ not a square.

This is valid.

If a quadrilateral fails to be a rectangle,

it cannot possibly be a square.

Compare Carefully

Suppose

$$A \rightarrow B.$$

Contrapositive:

$$\neg B \rightarrow \neg A$$

is equivalent to the original.

Converse:

$$B \rightarrow A$$

is not automatically equivalent.

This distinction must become automatic.

A Useful Table

Statement	Form	Equivalent to Original?
Original	$A \rightarrow B$	Yes
Converse	$B \rightarrow A$	Not necessarily
Contrapositive	$\neg B \rightarrow \neg A$	Yes

This table is worth understanding deeply.

Do Not Reason by Visual Similarity

The converse

$$B \rightarrow A$$

looks simpler than the contrapositive

$$\neg B \rightarrow \neg A.$$

But logical validity is not determined by visual simplicity.

TMUA Thinking

When evaluating an argument, ignore which statement *looks natural*.
Ask which statement is actually logically equivalent.

Valid Deduction 1

Suppose

$$A \rightarrow B$$

and

$$A.$$

Then

$$\boxed{B}.$$

This is valid.

In words:

A is true, and A implies $B \Rightarrow B$ is true.

Example

We know

$$12 \mid n \Rightarrow 4 \mid n.$$

We are told

$$12 \mid n.$$

Therefore

$$\boxed{4 \mid n}.$$

This is a legitimate forward use of the implication.

Valid Deduction 2

Suppose

$$A \rightarrow B$$

and

$$\neg B.$$

Then

$$\boxed{\neg A}.$$

Why?

Because the contrapositive is

$$\neg B \rightarrow \neg A.$$

This is logically equivalent to the original implication.

Example

Suppose

$$4 \mid n \Rightarrow n \text{ is even.}$$

We discover that

$$n \text{ is odd.}$$

Then n is not even.

Therefore, by the contrapositive,

$$\boxed{4 \nmid n.}$$

This deduction is valid.

Invalid Deduction 1

Suppose

$$A \rightarrow B$$

and

$$B.$$

Can we conclude A ?

No.

That would use the converse

$$B \rightarrow A,$$

which has not been established.

Common Mistake

Knowing the consequence occurred does not tell us which possible cause or condition produced it.

Example

We know

$$4 \mid n \Rightarrow n \text{ is even.}$$

Suppose

$$n \text{ is even.}$$

Can we conclude

$$4 \mid n?$$

No.

Take

$$n = 6.$$

The conclusion of the original implication is true,
but the hypothesis is false.

Invalid Deduction 2

Suppose

$$A \rightarrow B$$

and

$$\neg A.$$

Can we conclude

$$\neg B?$$

No.

This would amount to claiming

$$\neg A \rightarrow \neg B,$$

which does not follow from the original implication.

Example

We know

$$4 \mid n \Rightarrow n \text{ is even.}$$

Suppose

$$4 \nmid n.$$

Can we conclude that n is odd?

No.

Take

$$n = 6.$$

Then

$$4 \nmid 6,$$

but 6 is still even.

The Four Fundamental Patterns

Given

$$A \rightarrow B,$$

compare:

Information	Conclusion
A	B
$\neg B$	$\neg A$
B	No definite conclusion about A
$\neg A$	No definite conclusion about B

The first two are valid.

The last two are not.

Implication as Set Inclusion

Suppose $A(x)$ and $B(x)$ are properties.

If

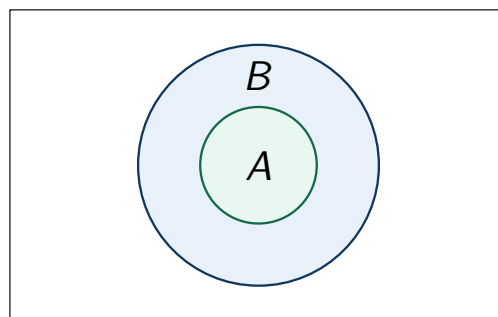
$$A(x) \rightarrow B(x),$$

then every object satisfying A also satisfies B .

Thus the set of A -objects lies inside the set of B -objects:

$$A \subseteq B.$$

Visualising the Original Implication



Every A -object is a B -object.

Therefore

$$A \rightarrow B.$$

Why the Converse Fails Visually

If

$$A \subseteq B,$$

there may be points in

$$B \setminus A.$$

Those points satisfy B ,
but not A .

Therefore

$$A \subseteq B$$

does not imply

$$B \subseteq A.$$

This is exactly why the converse may fail.

Why the Contrapositive Works Visually

If

$$A \subseteq B,$$

then anything outside B
must also be outside A .
Symbolically,

$$x \notin B \Rightarrow x \notin A.$$

That is precisely

$$\neg B \rightarrow \neg A.$$

So the set picture makes the contrapositive almost immediate.

Example 1: Inequalities

For real x ,

$$x > 5 \Rightarrow x > 2.$$

Converse:

$$x > 2 \Rightarrow x > 5$$

is false.

For example,

$$x = 3.$$

Contrapositive:

$$x \leq 2 \Rightarrow x \leq 5$$

is true.

Example 2: Powers

For real x ,

$$x > 2 \Rightarrow x^2 > 4.$$

The converse

$$x^2 > 4 \Rightarrow x > 2$$

is false because

$$x = -3$$

is a counterexample.

The contrapositive is

$$x^2 \leq 4 \Rightarrow x \leq 2.$$

This is true.

A Subtle Point About Negation

When forming the contrapositive,
the negation must be exact.
For example,

$$\neg(x > 2)$$

is

$$\boxed{x \leq 2},$$

not merely

$$x < 2.$$

Likewise,

$$\neg(x^2 > 4)$$

is

A Subtle Point About Negation (continued)

$$\boxed{x^2 \leq 4}.$$

Boundary cases matter.

Example 3: Divisibility

For integers n ,

$$10 \mid n \Rightarrow 5 \mid n.$$

Converse:

$$5 \mid n \Rightarrow 10 \mid n$$

is false.

Counterexample:

$$n = 15.$$

Contrapositive:

$$5 \nmid n \Rightarrow 10 \nmid n$$

is true.

Example 4: Functions

Suppose f is differentiable at $x = a$.

Then, under the standard real-variable framework,

$$f \text{ differentiable at } a \Rightarrow f \text{ continuous at } a.$$

The converse is not generally true:

$$\text{continuous} \not\Rightarrow \text{differentiable}.$$

For example,

$$f(x) = |x|$$

is continuous at 0 but not differentiable there.

Contrapositive of the Differentiability Result

Original:

differentiable at $a \Rightarrow$ continuous at a .

Contrapositive:

not continuous at $a \Rightarrow$ not differentiable at a .

This is valid.

Notice how useful the contrapositive can be:
to prove non-differentiability,
showing discontinuity is enough.

Sometimes Both Directions Hold

The converse is not *automatically* true.

But it can be true.

For example, for real x ,

$$x = 0 \Rightarrow x^2 = 0$$

and

$$x^2 = 0 \Rightarrow x = 0.$$

Therefore both directions hold.

From Two Implications to IFF

If both

$$A \rightarrow B$$

and

$$B \rightarrow A$$

are true,
then

$$A \text{ if and only if } B.$$

Symbolically,

$$A \leftrightarrow B.$$

Thus proving an “if and only if” statement normally requires establishing two directions.

Example

For an integer n ,

n is even

if and only if

n^2 is even.

One direction is

$$n \text{ even} \Rightarrow n^2 \text{ even.}$$

The reverse is

$$n^2 \text{ even} \Rightarrow n \text{ even.}$$

The reverse direction can be understood elegantly using a contrapositive.

Proving the Reverse by Contrapositive

We want to establish

$$n^2 \text{ even} \Rightarrow n \text{ even}.$$

Its contrapositive is

$$n \text{ odd} \Rightarrow n^2 \text{ odd}.$$

If

$$n = 2k + 1,$$

then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence n^2 is odd.

Therefore the original implication follows.

TMUA Example 1

Suppose $P \rightarrow Q$ is true.

Which statement must also be true?

(A) $Q \rightarrow P$

(B) $\neg P \rightarrow \neg Q$

(C) $\neg Q \rightarrow \neg P$

(D) $P \leftrightarrow Q$

The answer is

(C).

TMUA Example 2

We know

$$A \rightarrow B.$$

We are told that B is false.
Which conclusion is justified?

- (A) A is true
- (B) A is false
- (C) A may be true or false

By the contrapositive,

$$\neg B \rightarrow \neg A.$$

Therefore

TMUA Example 2 (continued)

(B).

TMUA Example 3

We know

$$A \rightarrow B.$$

We are told that A is false.

Which conclusion about B is justified?

TMUA Example 4

Claim:

If n^2 is divisible by 4, then n is divisible by 4.

Is this true?

Take

$$n = 2.$$

Then

$$n^2 = 4$$

is divisible by 4,
but 2 is not divisible by 4.
Therefore

$$n = 2$$

TMUA Example 4 (continued)

is a counterexample.
The claim is false.

TMUA Example 5

Suppose

$$A \rightarrow B$$

and

$$B \rightarrow C.$$

What can we conclude?

If A is true,
then B is true.

If B is true,
then C is true.

Therefore

$$\boxed{A \rightarrow C}.$$

This is an important chain of implications.

Chains of Reasoning

Suppose

$$A \rightarrow B,$$

$$B \rightarrow C,$$

and

$$C \rightarrow D.$$

Then

$$\boxed{A \rightarrow D.}$$

We can represent the chain as

$$A \longrightarrow B \longrightarrow C \longrightarrow D.$$

Paper 2 questions may hide such chains inside mathematical language.

Contrapositive of a Chain

If

$$A \rightarrow B \rightarrow C,$$

then

$$A \rightarrow C.$$

Its contrapositive is

$$\boxed{\neg C \rightarrow \neg A.}$$

We may also reason backwards through the chain:

$$\neg C \rightarrow \neg B \rightarrow \neg A.$$

Notice how the order reverses.

Faulty Argument 1

A student says:

All multiples of 8 are even.

14 is even.

Therefore

14 is a multiple of 8.

What went wrong?

The student reversed the implication.

$$8 \mid n \Rightarrow n \text{ even}$$

does not imply

$$n \text{ even} \Rightarrow 8 \mid n.$$

Faulty Argument 2

A student says:

If $x > 10$, then $x > 5$.

$$x \not> 10.$$

Therefore

$$x \not> 5.$$

This is invalid.

For example,

$$x = 7.$$

Then $x \not> 10$,

but $x > 5$.

The student incorrectly inferred

Faulty Argument 2 (continued)

$$\neg A \rightarrow \neg B.$$

Correct Version

From

$$x > 10 \Rightarrow x > 5,$$

the valid contrapositive is

$$x \leq 5 \Rightarrow x \leq 10.$$

Notice:

$$\neg B \rightarrow \neg A,$$

not

$$\neg A \rightarrow \neg B.$$

The order reverses.

Challenge 1

Write the converse and contrapositive of

$$x > 7 \Rightarrow x > 3.$$

Challenge 2

For integers n ,

$$9 \mid n \Rightarrow 3 \mid n.$$

Write the contrapositive.

Challenge 3

A statement $P \rightarrow Q$ is known to be true.
Which pair of truth values is impossible?

- (A) $P = T, Q = T$
- (B) $P = T, Q = F$
- (C) $P = F, Q = T$
- (D) $P = F, Q = F$

Challenge 4

Suppose

$$P \rightarrow Q$$

and

$$Q \rightarrow R.$$

We know R is false.
What can we conclude?
From

$$Q \rightarrow R,$$

the contrapositive gives

$$\neg R \rightarrow \neg Q.$$

Then from

Challenge 4 (continued)

$$P \rightarrow Q,$$

we obtain

$$\neg Q \rightarrow \neg P.$$

Therefore

$$\boxed{\neg P}.$$

Challenge 5

Suppose

$$P \rightarrow Q$$

and the converse

$$Q \rightarrow P$$

are both true.

What can we conclude?

$$\boxed{P \leftrightarrow Q}.$$

Thus P and Q have the same truth value.
Each is both enough to establish the other.

Do Not Memorise Arrows Blindly

The goal is not merely to remember:

$$A \rightarrow B$$

$$B \rightarrow A$$

$$\neg B \rightarrow \neg A.$$

Instead ask:

What does the original statement actually guarantee?

Then reason from that meaning.

A Useful Mental Model

Think of

$$A \rightarrow B$$

as saying:

There are no A -cases outside B .

Then:

$$\neg B \rightarrow \neg A$$

becomes obvious.

If something lies outside B ,
it cannot belong to A .

That is the contrapositive.

Counterexamples Build Mathematical Maturity

When confronted with a proposed implication,
do not immediately try to prove it.

First ask:

Could it be false?

Search for:

extreme cases

boundary cases

zero, negatives, small integers

and cases where the hypothesis barely holds.
This is a powerful TMUA habit.

Proof and Disproof Are Asymmetric

To establish a universal implication

$$A \rightarrow B,$$

you must justify it for every relevant case.

But to disprove it,

you need only one case with

$$A = T, \quad B = F.$$

Thus:

proof requires universality;

disproof may require only one counterexample.

Common Error 1: Converse = Original

Wrong assumption:

$$A \rightarrow B \implies B \rightarrow A.$$

Not valid.

The converse is a separate statement and requires separate justification.

Common Error 2: Wrong Contrapositive

Starting with

$$A \rightarrow B,$$

students sometimes write

$$\neg A \rightarrow \neg B.$$

This is not the contrapositive.

The correct contrapositive is

$$\boxed{\neg B \rightarrow \neg A.}$$

Negate **and** reverse.

Common Error 3: Incomplete Negation

Suppose

$$A: x > 4.$$

Then

$$\neg A$$

is

$$x \leq 4,$$

not merely

$$x < 4.$$

Likewise,

$$\neg(x \neq 3)$$

Common Error 3: Incomplete Negation (continued)

is

$$x = 3.$$

Precise negation matters.

Common Error 4: Invalid Counterexample

To disprove

$$A \rightarrow B,$$

you cannot choose a case where

$$A = F.$$

You specifically need

$$A = T, \quad B = F.$$

Always verify both parts of a proposed counterexample.

Common Error 5: Confusing Two Invalid Moves

Given

$$A \rightarrow B,$$

these are generally invalid:

$$B \therefore A$$

and

$$\neg A \therefore \neg B.$$

But these are valid:

$$A \therefore B$$

and

$$\neg B \therefore \neg A.$$

Keep the distinction sharp.

The Three Core Statements

Starting from

$$A \rightarrow B,$$

the converse is

$$B \rightarrow A,$$

while the contrapositive is

$$\neg B \rightarrow \neg A.$$

The fundamental equivalence is

$$A \rightarrow B \equiv \neg B \rightarrow \neg A.$$

The Key Distinction

$$\text{Original} \equiv \text{Contrapositive}$$

but

$$\text{Original} \not\equiv \text{Converse}$$

in general.

However, if both original and converse are true,
then we have

$$A \leftrightarrow B.$$

Valid Reasoning Map

Given

$$A \rightarrow B,$$

we may reason:

$$\boxed{A \Rightarrow B}$$

or

$$\boxed{\neg B \Rightarrow \neg A.}$$

But we may not automatically reason:

$$B \Rightarrow A$$

or

$$\neg A \Rightarrow \neg B.$$

This single distinction prevents many logical errors.

Self-Check

Before moving on, make sure you can:

- ➊ Define the converse of an implication.
- ➋ Define the contrapositive.
- ➌ Explain why the converse need not be true.
- ➍ Construct a valid counterexample.
- ➎ Explain why the contrapositive is equivalent to the original.
- ➏ Negate inequalities correctly.
- ➐ Recognise valid and invalid deductions.
- ➑ Chain implications together.
- ➒ Use the contrapositive in mathematical reasoning.
- ➓ Distinguish proof from disproof.

The TMUA Habit

Whenever you see

$$A \rightarrow B,$$

mentally generate:

$$\text{Original: } A \rightarrow B$$

$$\text{Converse: } B \rightarrow A$$

$$\text{Contrapositive: } \neg B \rightarrow \neg A$$

and remember:

$$\text{Original} \equiv \text{Contrapositive.}$$

Coming in Part 4

Necessary and Sufficient Conditions

What does it mean to say that a condition is

sufficient

for another?

What does it mean to say that a condition is

necessary?

And how are these connected to

if

only if

if and only if?

This is where logical language becomes especially important for TMUA Paper 2.

**Do Not Reverse an Implication.
Use the Contrapositive.
Search for Counterexamples.**

Mathematics Elevate Academy
Rishabh Kumar

TMUA Paper 2 Masterclass

Part 4: Necessary and Sufficient Conditions Understanding Exactly What a Condition Guarantees

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From Implication to Conditions

In the previous parts, we studied

$$\boxed{A \rightarrow B}.$$

We learned that:

$$A \rightarrow B$$

does not generally imply

$$B \rightarrow A,$$

but it is logically equivalent to its contrapositive

$$\neg B \rightarrow \neg A.$$

We now give another interpretation to the same implication.

The Same Arrow, New Language

If

$$A \rightarrow B,$$

then we may say:

A is sufficient for B

and

B is necessary for A .

These are two descriptions of the same logical relationship.

The Central Idea

Suppose

$$A \rightarrow B.$$

Then A guarantees B .

Therefore A is

sufficient

for B .

At the same time, A cannot occur without B .

Therefore B is

necessary

for A .

Part 4 Roadmap

We will study:

- ① Sufficient conditions
- ② Necessary conditions
- ③ The direction of implication
- ④ “If” and “only if”
- ⑤ Necessary but not sufficient
- ⑥ Sufficient but not necessary
- ⑦ Necessary and sufficient conditions
- ⑧ Counterexamples
- ⑨ Set interpretations
- ⑩ TMUA-style reasoning

What Does Sufficient Mean?

Suppose A is sufficient for B .
This means:

A being true is enough to guarantee B .

Therefore

$A \text{ sufficient for } B \iff A \rightarrow B.$

Think:

sufficient = enough.

Example: Divisibility

For an integer n ,

$$4 \mid n \Rightarrow 2 \mid n.$$

Therefore:

Being divisible by 4 is sufficient for being even.

If we know $4 \mid n$,
we already have enough information to conclude that n is even.

But Sufficient Does Not Mean Necessary

Being divisible by 4 is sufficient for being even.

But is it necessary?

No.

For example,

$$n = 6$$

is even,
but

$$4 \nmid 6.$$

Therefore divisibility by 4 is

sufficient but not necessary

for being even.

Another Sufficient Condition

For real x ,

$$x > 10 \Rightarrow x > 5.$$

Therefore

$$x > 10 \text{ is sufficient for } x > 5.$$

But $x > 10$ is not necessary.

For example,

$$x = 7$$

satisfies $x > 5$ without satisfying $x > 10$.

Sufficient Means “Enough,” Not “Required”

If A is sufficient for B ,
then A is one way of guaranteeing B .
There may be many other ways for B to be true.
Thus

$$A \rightarrow B$$

does not tell us

$$B \rightarrow A.$$

Key Idea

A sufficient condition guarantees the conclusion, but it need not be the only way in which the conclusion can occur.

What Does Necessary Mean?

Suppose B is necessary for A .

This means:

A cannot be true unless B is true.

Therefore

B necessary for $A \iff A \rightarrow B$.

Think:

necessary = required.

Example: Divisibility

If an integer is divisible by 4,
then it must be even.

Thus:

$$4 \mid n \Rightarrow 2 \mid n.$$

So being even is

necessary for being divisible by 4.

You cannot have a multiple of 4 that is not even.

Necessary Does Not Mean Sufficient

Being even is necessary for being divisible by 4.
But is being even enough to guarantee divisibility by 4?
No.
For example,

$$n = 6.$$

Thus being even is

necessary but not sufficient

for divisibility by 4.

Example: Geometry

For a quadrilateral to be a square,
being a rectangle is necessary.
Why?

square $\rightarrow$ rectangle.

But being a rectangle is not sufficient for being a square.
A rectangle can have unequal adjacent side lengths.
Therefore:

rectangle is necessary but not sufficient for square.

The Rule You Must Know

If

$$A \rightarrow B,$$

then

A is sufficient for B

and

B is necessary for A .

Notice the reversal of language:

$$A \longrightarrow B$$

sufficient $\longrightarrow$ necessary.

A Memory Aid

For

$$A \rightarrow B,$$

think:

A is enough for B

and

B is required for A .

Thus:

enough $\rightarrow$ required.

Do not memorise the words without understanding the arrow.

Translation Table

Statement	Logical Form
A is sufficient for B	$A \rightarrow B$
B is necessary for A	$A \rightarrow B$
A only if B	$A \rightarrow B$
B if A	$A \rightarrow B$

All four rows describe the same implication.

“If” and Sufficient

Consider

B if A .

This means

$A \rightarrow B$.

Therefore A is sufficient for B .

So:

$$B \text{ if } A \iff A \text{ sufficient for } B.$$

“Only If” and Necessary

Consider

A only if B .

This means

$$A \rightarrow B.$$

Therefore B is necessary for A .

So:

$$A \text{ only if } B \iff B \text{ necessary for } A.$$

A Very Important Translation

The statement

A is sufficient for B

means

B if A .

The statement

A is necessary for B

means

B only if A .

These two statements point in opposite directions.

The Most Common Confusion

Compare:

A is sufficient for B

with

A is necessary for B .

The first means

$$A \rightarrow B.$$

The second means

$$B \rightarrow A.$$

The same words A and B appear,
but the logical arrows are reversed.

Necessary but Not Sufficient

Suppose A is necessary for B .
Then

$$B \rightarrow A.$$

But suppose A is not sufficient for B .
Then

$$A \not\rightarrow B.$$

Together:

$$B \rightarrow A, \quad A \not\rightarrow B.$$

Example

Consider:

B : n is divisible by 4,

A : n is even.

We have

$$B \rightarrow A.$$

Therefore being even is necessary for divisibility by 4.

But

$$A \not\rightarrow B.$$

For example,

$$n = 6.$$

So being even is necessary but not sufficient.

Another Example: Squares

Let

B : quadrilateral is a square,

A : quadrilateral is a rectangle.

Then

$$B \rightarrow A.$$

But

$$A \not\rightarrow B.$$

Thus being a rectangle is

necessary but not sufficient

for being a square.

Sufficient but Not Necessary

Suppose A is sufficient for B .
Then

$$A \rightarrow B.$$

But suppose A is not necessary for B .
Then

$$B \not\rightarrow A.$$

Together:

$$A \rightarrow B, \quad B \not\rightarrow A.$$

Example

Let

$A :$ n is divisible by 4,

$B :$ n is even.

Then

$$A \rightarrow B.$$

But

$$B \not\rightarrow A.$$

Again,

$$n = 6$$

provides a counterexample to the reverse implication.
Thus divisibility by 4 is sufficient but not necessary for being even.

When Both Directions Hold

Suppose

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Then A is sufficient for B ,
and A is necessary for B .
Therefore A is

necessary and sufficient for B .

Logical Form

If A is necessary and sufficient for B ,
then

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Therefore

$$A \leftrightarrow B.$$

This is exactly the meaning of

A if and only if B .

Three Equivalent Ways to Say It

These statements mean the same thing:

A is necessary and sufficient for B

A if and only if B

$A \leftrightarrow B$.

And each means:

$(A \rightarrow B) \text{ and } (B \rightarrow A)$.

Example: Zero

For real x ,

$$x = 0$$

is necessary and sufficient for

$$x^2 = 0.$$

Why?

First,

$$x = 0 \Rightarrow x^2 = 0.$$

Second,

$$x^2 = 0 \Rightarrow x = 0.$$

Therefore

Example: Zero (continued)

$$x = 0 \leftrightarrow x^2 = 0.$$

Example: Even Integers

For integers n ,

n is even

if and only if there exists an integer k such that

$$n = 2k.$$

Thus

$$n \text{ even} \leftrightarrow n = 2k \text{ for some } k \in \mathbb{Z}.$$

This is not merely a useful implication.
It characterises even integers exactly.

A Characterisation

A necessary-and-sufficient condition does something stronger than a one-way implication.
It gives an exact characterisation.
If

$$A \leftrightarrow B,$$

then the objects satisfying A
are exactly the objects satisfying B .
There are no extra cases in either direction.

Set Interpretation

If

$$A \rightarrow B,$$

then

$$A \subseteq B.$$

If also

$$B \rightarrow A,$$

then

$$B \subseteq A.$$

Therefore

$$\boxed{A = B}$$

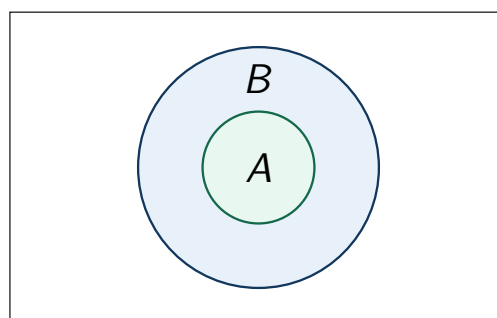
Set Interpretation (continued)

as sets of cases.

This is the set interpretation of

$$A \leftrightarrow B.$$

Visualising Sufficient but Not Necessary



Here

$$A \subsetneq B.$$

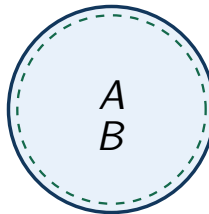
So A is sufficient for B ,
but not necessary.

Visualising Necessary and Sufficient

If

$$A \leftrightarrow B,$$

the two sets coincide.



The A -cases and B -cases are exactly the same.

How to Test Sufficiency

Suppose someone claims:

$$A \text{ is sufficient for } B.$$

Translate it:

$$A \rightarrow B.$$

To disprove it,
find a case where

$$A = T, \quad B = F.$$

One such counterexample is enough.

How to Test Necessity

Suppose someone claims:

A is necessary for B .

Translate it:

$$B \rightarrow A.$$

To disprove it,
find a case where

$$B = T, \quad A = F.$$

Notice that the order has changed.

A Powerful Exam Strategy

Do not reason directly from the words

necessary and sufficient.

First translate them into arrows.

For example:

A necessary for B

becomes

$$B \rightarrow A.$$

Then analyse the implication.

TMUA Thinking

Translate language into logic before doing mathematics.

Example 1: Inequalities

Consider the conditions

$$A : x > 10$$

and

$$B : x > 5.$$

We have

$$A \rightarrow B.$$

Therefore A is sufficient for B ,
and B is necessary for A .
But the converse fails.
Thus:

A sufficient but not necessary for B .

Example 2: Positive Numbers

For real x , consider

$$A : x > 0$$

and

$$B : x^2 > 0.$$

Certainly,

$$x > 0 \Rightarrow x^2 > 0.$$

So A is sufficient for B .
But is A necessary?
No.
For example,

$$x = -2$$

Example 2: Positive Numbers (continued)

also gives

$$x^2 > 0.$$

Example 3: Square Root Condition

For real x , consider

$$A : x = 4$$

and

$$B : \sqrt{x} = 2.$$

We have

$$x = 4 \Rightarrow \sqrt{x} = 2.$$

And

$$\sqrt{x} = 2 \Rightarrow x = 4.$$

Thus

$x = 4$ is necessary and sufficient for $\sqrt{x} = 2$.
--

Example 4: Geometry

Consider a quadrilateral.

Let

A : it is a square,

B : it has four equal sides.

We know

$$A \rightarrow B.$$

So having four equal sides is necessary for being a square.

But is it sufficient?

No.

A non-square rhombus has four equal sides.

Therefore B is necessary but not sufficient for A .

Example 5: Quadratic Roots

Consider a quadratic

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

Let

$$A : b^2 - 4ac = 0,$$

and

B : the quadratic has a repeated real root.

For real coefficients,

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Example 5: Quadratic Roots (continued)

Therefore

$$b^2 - 4ac = 0$$

is necessary and sufficient for a repeated real root.

Necessary Conditions and the Contrapositive

Suppose B is necessary for A .

Then

$$A \rightarrow B.$$

The contrapositive gives

$$\neg B \rightarrow \neg A.$$

This provides another way to understand necessity.

If a required condition fails,
the original condition is impossible.

Example

Being even is necessary for being divisible by 4.

Thus:

$$4 \mid n \rightarrow n \text{ even.}$$

Contrapositive:

$$n \text{ not even} \rightarrow 4 \nmid n.$$

For integers:

$$n \text{ odd} \rightarrow 4 \nmid n.$$

If a necessary condition fails, the target condition cannot hold.

A Useful Problem-Solving Principle

Suppose you want to determine whether A is possible.

If B is necessary for A ,

then showing

$$\neg B$$

immediately gives

$$\neg A.$$

So necessary conditions can be powerful tools for ruling out possibilities.

One Property May Have Many Necessary Conditions

Suppose A implies each of

$B, \quad C, \quad D.$

Then

$$A \rightarrow B,$$

$$A \rightarrow C,$$

and

$$A \rightarrow D.$$

Therefore B, C, D are all necessary conditions for A .
But none of them individually need be sufficient.

Example: Square

If a quadrilateral is a square, then:

it is a rectangle,

it is a rhombus,

its diagonals are equal,

and

its diagonals are perpendicular.

These provide necessary properties of a square.
But a single one of them does not by itself characterise a square.

Combining Conditions

Sometimes no single condition is sufficient,
but several conditions together are.
Suppose

A and B

together imply C .
Then

$$(A \wedge B) \rightarrow C.$$

This is different from saying

$$A \rightarrow C$$

or

$$B \rightarrow C.$$

The combination may be what provides sufficiency.

TMUA Example 1

Suppose P is sufficient for Q .
Which statement must be true?

(A) $Q \rightarrow P$

(B) $P \rightarrow Q$

(C) $\neg P \rightarrow \neg Q$

(D) $P \leftrightarrow Q$

The answer is

$$(B).$$

TMUA Example 2

Suppose P is necessary for Q .
Translate this into implication notation.
Since Q cannot occur without P ,

$$Q \rightarrow P.$$

Therefore Q is sufficient for P ,
while P is necessary for Q .

TMUA Example 3

Suppose P is sufficient but not necessary for Q .
Which relationships hold?

$$P \rightarrow Q$$

but

$$Q \not\Rightarrow P.$$

Therefore there must be at least one possible case where

$$Q = T, \quad P = F.$$

TMUA Example 4

Suppose P is necessary but not sufficient for Q .
Then:

$$Q \rightarrow P$$

but

$$P \not\Rightarrow Q.$$

Therefore there must be a possible case where

$$P = T, \quad Q = F.$$

TMUA Example 5

Suppose P is both necessary and sufficient for Q .
Which statement follows?

(A) $P \rightarrow Q$

only?

(B) $Q \rightarrow P$

only?

No.

Both directions hold:

$$P \leftrightarrow Q.$$

TMUA Example 6

For real x , consider:

$$A : x > 3,$$

$$B : x^2 > 9.$$

Is A sufficient for B ?

Yes:

$$x > 3 \Rightarrow x^2 > 9.$$

Is A necessary for B ?

No.

For example,

$$x = -4$$

gives

TMUA Example 6 (continued)

$$x^2 = 16 > 9.$$

Thus A is sufficient but not necessary.

Problem: Combining Information

Suppose:

A is sufficient for B

and

B is sufficient for C .

Then

$$A \rightarrow B$$

and

$$B \rightarrow C.$$

Therefore

$$\boxed{A \rightarrow C.}$$

Hence A is sufficient for C .

What About Necessity?

From

$$A \rightarrow B \rightarrow C,$$

we know:

C is necessary for B ,

and

B is necessary for A .

Therefore C is also necessary for A .

So:

$$\boxed{A \rightarrow C}$$

can be read in both ways:

What About Necessity? (continued)

A sufficient for C ,

C necessary for A .

A Chain of Conditions

Suppose

$$A \rightarrow B \rightarrow C \rightarrow D.$$

Then:

A is sufficient for B, C, D ,

while

D is necessary for A, B, C

whenever the corresponding implication chain reaches D .

In particular,

$$\boxed{A \rightarrow D.}$$

So A is sufficient for D ,
and D is necessary for A .

Disproving “Sufficient”

Claim:

A is sufficient for B .

Translation:

$$A \rightarrow B.$$

To disprove:

Find $A = T, B = F$.

Example:

$x^2 > 4$ is sufficient for $x > 2$.

Counterexample:

$$x = -3.$$

Disproving “Necessary”

Claim:

A is necessary for B .

Translation:

$$B \rightarrow A.$$

To disprove:

Find $B = T, A = F$.

Example:

$x > 0$ is necessary for $x^2 > 0$.

Counterexample:

$$x = -1.$$

Disproving “Necessary and Sufficient”

Claim:

$$A \leftrightarrow B.$$

There are two ways it can fail.
Either find

$$A = T, B = F$$

or find

$$B = T, A = F.$$

A biconditional requires both directions.
Breaking either direction is enough to disprove it.

Necessary Is Not the Same as Important

In ordinary language, “necessary” may sound like

important

or

useful.

That is not its mathematical meaning.
Mathematically,

A necessary for B

means exactly

$$B \rightarrow A.$$

Nothing more and nothing less.

Sufficient Is Not the Same as Complete

A sufficient condition need not describe every possible case.

If

$$A \rightarrow B,$$

then A guarantees B .

But there may be many B -cases outside A .

So sufficiency means:

enough to guarantee

not

the only possibility.

Conditions Are About Direction

The words can feel difficult.

The arrows are simpler.

Whenever you see:

necessary,

sufficient,

only if,

if,

translate the sentence into

$\rightarrow$.

Then reason mathematically.

Ask More Than “Is It True?”

Suppose you discover that

$$A \rightarrow B$$

is true.

Do not stop.

Ask:

Is the converse also true?

If yes,

$$A \leftrightarrow B.$$

If no,

you have only a one-way relationship.

This distinction is fundamental to mathematical maturity.

A Hierarchy of Understanding

At a basic level:

Know a theorem.

At a stronger level:

Know what the theorem guarantees.

At a deeper level:

Know which assumptions are necessary, which are sufficient, and why.

This deeper level of reasoning is particularly valuable in university mathematics and admissions tests.

Challenge 1

For integers n , consider

$$A : 12 \mid n,$$

$$B : 3 \mid n.$$

Classify A relative to B .

Challenge 1 (continued)

A is sufficient but not necessary for B .

Challenge 2

For real x , consider

$$A : x = 2,$$

$$B : x^2 = 4.$$

Is A sufficient for B ?

Yes.

Is A necessary for B ?

No, because

$$x = -2$$

also satisfies B .

Therefore

A is sufficient but not necessary for B .

Challenge 3

For real x , let

$$A : x = 0,$$

$$B : x^4 = 0.$$

Then

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Therefore

A is necessary and sufficient for B .

Challenge 4

Suppose:

A is necessary for B ,

and

B is necessary for C .

What follows?

Translate:

$$B \rightarrow A,$$

$$C \rightarrow B.$$

Therefore

$$C \rightarrow B \rightarrow A.$$

Challenge 4 (continued)

Hence

A is necessary for C .

Challenge 5

Suppose A is sufficient for B ,
and B is necessary for A .
Are these two pieces of information different?

Challenge 6

Suppose:

A is sufficient for B ,

and

A is necessary for B .

Then

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

Therefore

$$A \leftrightarrow B.$$

So A is necessary and sufficient for B .

Common Error 1: Reversing Necessity

Wrong:

A necessary for $B \Rightarrow A \rightarrow B$.

Correct:

A necessary for $B \Rightarrow B \rightarrow A$.

If B happens,
the necessary condition A must be present.

Common Error 2: Reversing Sufficiency

Wrong:

A sufficient for $B \Rightarrow B \rightarrow A$.

Correct:

A sufficient for $B \Rightarrow A \rightarrow B$.

A provides enough information to conclude B .

Common Error 3: Necessary = Sufficient

A condition may be necessary without being sufficient.
A condition may be sufficient without being necessary.
These are different ideas.
Only when both directions hold do we have

necessary and sufficient.

Common Error 4: Forgetting the Domain

Whether a condition is necessary or sufficient may depend on the domain.
For example, statements about

$$x^2$$

can behave differently depending on whether x is:

$$\mathbb{R}, \quad \mathbb{Z}, \quad \mathbb{N},$$

or another specified set.

Common Mistake

Always identify the domain before testing a logical condition.

Common Error 5: Checking Only One Direction

To establish

A necessary and sufficient for B ,

you must establish

$$A \rightarrow B$$

and

$$B \rightarrow A.$$

One direction alone establishes only necessity or sufficiency, not both.

The Central Rule

Everything in this lesson can be built from

$$A \rightarrow B.$$

This means:

A is sufficient for B

and simultaneously

B is necessary for A .

That is the central relationship.

The Complete Translation Table

Language	Logical Form
If A then B	$A \rightarrow B$
B if A	$A \rightarrow B$
A only if B	$A \rightarrow B$
A sufficient for B	$A \rightarrow B$
B necessary for A	$A \rightarrow B$
A iff B	$A \leftrightarrow B$
A necessary and sufficient for B	$A \leftrightarrow B$

Four Possibilities

Relative to B , a condition A may be:

sufficient but not necessary

necessary but not sufficient

necessary and sufficient

or

neither necessary nor sufficient.

TMUA questions may require you to distinguish these carefully.

Self-Check

Before moving on, make sure you can explain:

- ① What does “sufficient” mean?
- ② What does “necessary” mean?
- ③ If $A \rightarrow B$, which condition is sufficient?
- ④ If $A \rightarrow B$, which condition is necessary?
- ⑤ How does “if” relate to sufficiency?
- ⑥ How does “only if” relate to necessity?
- ⑦ What does necessary and sufficient mean?
- ⑧ How do you disprove sufficiency?
- ⑨ How do you disprove necessity?
- ⑩ Why does the domain matter?

The TMUA Habit

When you see a statement involving

necessary or sufficient,

do three things:

1. Identify the two conditions

2. Translate into an implication

3. Test the direction carefully

Do not let the wording replace the mathematics.

Coming in Part 5

Quantifiers and Mathematical Statements

We will study the meaning of

for all

and

there exists.

We will then learn how to negate statements such as

For every x , $P(x)$

Coming in Part 5 (continued)

and

There exists x such that $P(x)$.

This leads directly into counterexamples, proof statements, and deeper TMUA Paper 2 reasoning.

Sufficient Means Enough. Necessary Means Required. Necessary + Sufficient Means Exact.

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 5: Quantifiers, Negation and Counterexamples
For All, There Exists, and the Logic of Mathematical Claims

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From Conditions to Mathematical Claims

So far we have studied statements such as

$$A \rightarrow B$$

and

$$A \leftrightarrow B.$$

But mathematical statements frequently say something about **many possible values**.
For example:

Every even integer has an even square.

or

There exists a real number whose square is 2.

These statements involve **quantifiers**.

Two Fundamental Quantifiers

The two central quantifiers are:

$$\forall$$

meaning

for every / for all

and

$$\exists$$

meaning

there exists / for some .

Much of mathematical reasoning depends on distinguishing these correctly.

Why Quantifiers Matter

Compare:

Every real number satisfies $x^2 \geq 0$

with

Some real number satisfies $x^2 = 4$.

The first makes a claim about

all real numbers,

while the second requires only

one suitable real number.

The logical burden is completely different.

Part 5 Roadmap

We will study:

- 1 Open statements and domains
- 2 Universal quantification: $\forall$
- 3 Existential quantification: $\exists$
- 4 How to prove universal statements
- 5 How to disprove universal statements
- 6 Counterexamples
- 7 How to prove existential statements
- 8 Negating quantified statements
- 9 Multiple and nested quantifiers
- 10 Why quantifier order matters
- 11 TMUA-style reasoning

A Statement Depending on a Variable

Consider

$$x^2 > 4.$$

Is this true or false?

We cannot answer until we know x .

For example,

$$x = 3$$

makes it true,
while

$$x = 1$$

makes it false.

So $x^2 > 4$ is not yet a complete proposition about a specified x .

Predicate Notation

We may write

$$P(x) : x^2 > 4.$$

Here $P(x)$ is a property depending on x .

Once we quantify x , we obtain statements such as

$$\forall x, P(x)$$

or

$$\exists x, P(x).$$

These are complete mathematical claims.

The Domain Matters

Consider

$$\exists x, \quad x^2 = 2.$$

If the domain is

$$\mathbb{R},$$

the statement is true.

If the domain is

$$\mathbb{Z},$$

the statement is false.

Thus the meaning of a quantified statement depends critically on its domain.

Common Mistake

Never analyse a quantified mathematical statement while ignoring its domain.

The Universal Quantifier

The notation

$$\forall x, \quad P(x)$$

means

$$P(x) \text{ is true for every } x$$

in the specified domain.

The symbol

$$\forall$$

is read as

$$\text{for all}$$

or

The Universal Quantifier (continued)

for every .

Example

For real x ,

$$\forall x \in \mathbb{R}, \quad x^2 \geq 0.$$

This means:

every real number has a non-negative square.

The statement must hold for

$$x = 0,$$

positive x ,
negative x ,
and every other real number.

Universal Means No Exceptions

To say

$$\forall x, P(x)$$

is to say that there are

no exceptions.

Every allowed x must satisfy $P(x)$.
Therefore even one value a for which

$$P(a)$$

is false destroys the universal claim.

Example of a False Universal Claim

Consider

$$\forall x \in \mathbb{R}, \quad x^2 > x.$$

Try

$$x = 1.$$

Then

$$1^2 > 1$$

becomes

$$1 > 1,$$

which is false.

Therefore the universal statement is false.

Counterexamples Revisited

Suppose someone claims

$$\boxed{\forall x, P(x)}.$$

To disprove the statement,
we need only find one value a such that

$$\boxed{\neg P(a)}.$$

That value is a

$$\boxed{\text{counterexample}}.$$

One Counterexample Is Enough

Claim:

$$\boxed{\forall n \in \mathbb{Z}, \quad n^2 > n.}$$

Take

$$n = 0.$$

Then

$$0^2 > 0$$

is false.

Therefore the entire universal claim is false.

We do not need to test every integer.

Why One Example Cannot Prove a Universal Claim

Suppose we test:

$$1^2 \geq 1,$$

$$2^2 \geq 2,$$

$$3^2 \geq 3,$$

$$4^2 \geq 4.$$

These examples may suggest a pattern.
But they do not prove

$$\forall n \in \mathbb{N}, \quad n^2 \geq n.$$

A finite number of examples cannot generally establish an infinite universal claim.

Examples Can Suggest; Proof Must Establish

Examples are valuable for:

discovering patterns

testing conjectures

finding counterexamples.

But a universal theorem requires reasoning that applies to every case in the stated domain.

Mathematical Maturity

Evidence is not the same thing as proof.

The Arbitrary Element Method

Suppose we want to prove

$$\forall x \in D, \quad P(x).$$

A standard strategy is:

Let x be an arbitrary element of D .

Then prove

$$P(x)$$

without imposing any additional special restriction on x .

Because x was arbitrary, the argument applies to every element of D .

Example

Prove:

$$\forall n \in \mathbb{Z}, \quad n \text{ even} \Rightarrow n^2 \text{ even.}$$

Let n be an arbitrary even integer.

Then

$$n = 2k$$

for some $k \in \mathbb{Z}$.

Hence

$$n^2 = 4k^2 = 2(2k^2).$$

Therefore n^2 is even.

Why “Arbitrary” Matters

Suppose we want to prove something for every integer.
Choosing

$$n = 8$$

and proving the result for 8 proves only one case.

Instead, we choose an arbitrary integer n .

Our argument must use only properties available to every integer satisfying the hypothesis.

That is what makes the proof universal.

The Existential Quantifier

The notation

$$\exists x, P(x)$$

means

there is at least one x for which $P(x)$ is true.

The symbol

$$\exists$$

is read as

there exists .

Example

Consider

$$\boxed{\exists x \in \mathbb{R} \text{ such that } x^2 = 9.}$$

This is true.
For example,

$$x = 3$$

satisfies the equation.
Indeed,

$$x = -3$$

does too.
But only one example is required to establish existence.

Existence Requires At Least One Witness

To prove

$$\exists x, P(x),$$

it is often enough to exhibit one suitable value a and show

$$P(a).$$

Such an a is sometimes called a

witness.

Thus existential claims and universal claims behave very differently.

Universal vs Existential

Statement	Requirement
$\forall x, P(x)$	$P(x)$ must hold for every x
$\exists x, P(x)$	$P(x)$ must hold for at least one x

For a universal claim,

one failure is enough to disprove it.

For an existential claim,

one success is enough to prove it.

What Is the Negation of “Every”?

Suppose the statement is

$$\forall x, P(x).$$

What does it mean for this statement to be false?

It means that $P(x)$ does not hold for every x .

Therefore there must be at least one x for which $P(x)$ fails.

Thus:

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x).$$

In Words

The negation of

Every x has property P

is

There exists at least one x that does not have property P .

It is **not**

No x has property P .

That would be much stronger.

Example

Statement:

$$\forall x \in \mathbb{R}, \quad x^2 > 0.$$

Its negation is:

$$\exists x \in \mathbb{R} \text{ such that } x^2 \leq 0.$$

Indeed,

$$x = 0$$

satisfies

$$x^2 = 0.$$

So the original universal statement is false.

Negating the Property Correctly

Suppose

$$P(x) : x > 5.$$

Then

$$\neg P(x)$$

is

$$x \leq 5.$$

Therefore

$$\neg(\forall x, x > 5)$$

becomes

$$\exists x \text{ such that } x \leq 5.$$

The inequality must be negated precisely.

What Is the Negation of “There Exists”?

Suppose

$$\exists x, P(x).$$

For this statement to be false,
there must be no x satisfying $P(x)$.
Therefore every x must fail P .
Hence:

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x).$$

In Words

The negation of

There exists an x with property P

is

Every x fails property P .

Equivalently:

There is no x with property P .

Example

Statement:

$\exists x \in \mathbb{R}$ such that $x^2 = -1$.

Its negation is

$\forall x \in \mathbb{R}, x^2 \neq -1$.

Since every real square is non-negative,
the negation is true.

Therefore the original existential statement is false.

The Fundamental Rule

When a negation passes through a quantifier,
the quantifier changes:

$$\boxed{\forall \longleftrightarrow \exists.}$$

Specifically,

$$\boxed{\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)}$$

and

$$\boxed{\neg(\exists x, P(x)) \equiv \forall x, \neg P(x).}$$

A Useful Memory Structure

Think:

$$\boxed{\text{NOT ALL} = \text{AT LEAST ONE NOT}}$$

and

$$\boxed{\text{NOT EXISTS} = \text{ALL NOT.}}$$

Or symbolically:

$$\neg \forall \rightsquigarrow \exists \neg$$

$$\neg \exists \rightsquigarrow \forall \neg.$$

Do Not Negate Only the Quantifier

For example,

$$\neg(\forall x, P(x))$$

does not become merely

$$\exists x, P(x).$$

The property must also be negated.

Correct:

$$\exists x, \neg P(x).$$

Both changes are required.

Translating “Every”

Consider:

Every integer divisible by 6 is divisible by 3.

This can be written as

$$\forall n \in \mathbb{Z}, \quad 6 \mid n \Rightarrow 3 \mid n.$$

Notice that a universal quantifier and an implication appear together.

Negating the Whole Statement

Original:

$$\forall n \in \mathbb{Z}, \quad 6 \mid n \Rightarrow 3 \mid n.$$

To negate it:

$$\exists n \in \mathbb{Z}$$

such that the implication fails.

An implication fails when

$$6 \mid n$$

is true and

$$3 \nmid n$$

is false.

Thus the negation is

Negating the Whole Statement (continued)

$$\boxed{\exists n \in \mathbb{Z} : 6 \mid n \text{ and } 3 \nmid n.}$$

A General Pattern

The statement

$$\forall x, \quad P(x) \rightarrow Q(x)$$

is negated as

$$\exists x, \quad \neg(P(x) \rightarrow Q(x)).$$

But

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q.$$

Therefore:

$$\neg[\forall x, P(x) \rightarrow Q(x)] \equiv \exists x, P(x) \wedge \neg Q(x).$$

This Is Exactly a Counterexample

A universal implication says:

Every P -case is a Q -case.

Its negation says:

There exists a P -case that is not a Q -case.

That is precisely the definition of a counterexample.

So counterexamples are not an isolated trick.

They arise naturally from logical negation.

Existence Is Not Uniqueness

Consider

$$\exists x, \quad x^2 = 4.$$

This says at least one solution exists.

In fact,

$$x = 2$$

and

$$x = -2$$

both work.

So existence alone does not imply uniqueness.

Exactly One

Sometimes mathematics requires:

There exists exactly one x satisfying $P(x)$.

This is often denoted

$$\exists! x, \quad P(x).$$

It contains two ideas:

existence

and

uniqueness.

Example of Unique Existence

Over the real numbers,

$$x^3 = 8$$

has exactly one solution:

$$x = 2.$$

Thus:

$$\exists! x \in \mathbb{R} \text{ such that } x^3 = 8.$$

By contrast,

$$x^2 = 4$$

does not have a unique real solution.

Statements Can Have More Than One Quantifier

Consider:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } y > x.$$

This says:

for every real x ,

we can find some real y larger than x .

This statement is true.

For example, given x , choose

$$y = x + 1.$$

Reverse the Quantifiers

Now consider:

$$\boxed{\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y > x.}$$

This says:

there is one real number y

that is larger than every real number x .

This is false.

No largest real number exists.

Quantifier Order Matters

Compare:

$$\boxed{\forall x \exists y, y > x}$$

with

$$\boxed{\exists y \forall x, y > x.}$$

The first is true over $\mathbb{R}$.

The second is false.

Thus:

$$\boxed{\forall x \exists y \neq \exists y \forall x}$$

in general.

Why the Difference?

In

$$\forall x \exists y,$$

the value of y may depend on x .
For example,

$$y = x + 1.$$

But in

$$\exists y \forall x,$$

we must choose one fixed y
that works for every x .
That is a much stronger requirement.

Another Example

Consider:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x + y = 0.$$

This is true.
Given x , choose

$$\boxed{y = -x}.$$

But:

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad x + y = 0$$

is false.
No single fixed y works for every x .

Negate One Quantifier at a Time

Suppose

$$\forall x \exists y, \quad P(x, y).$$

Its negation is

$$\neg(\forall x \exists y, \quad P(x, y)).$$

Move the negation inward:

$$\exists x \neg(\exists y, \quad P(x, y)).$$

Then:

$$\boxed{\exists x \forall y, \quad \neg P(x, y).}$$

The General Pattern

$$\boxed{\neg(\forall x \exists y \quad P(x, y)) \equiv \exists x \forall y \neg P(x, y)}$$

Similarly,

$$\boxed{\neg(\exists x \forall y \quad P(x, y)) \equiv \forall x \exists y \neg P(x, y).}$$

Every quantifier switches as the negation passes through it.

Example

Original:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y > x.$$

Negation:

$$\boxed{\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y \leq x.}$$

In words:

there exists a real number x such that every real number y is at most x .

That would mean x is a largest real number.

No such x exists.

TMUA Example 1

Which is the correct negation of

$$\forall x \in \mathbb{R}, P(x)?$$

(A) $\forall x, \neg P(x)$

(B) $\exists x, P(x)$

(C) $\exists x, \neg P(x)$

(D) $\neg \exists x, P(x)$

The answer is

$$\boxed{(C)}.$$

TMUA Example 2

Negate:

$$\boxed{\exists n \in \mathbb{Z}, \quad n^2 = 2.}$$

The negation is:

$$\boxed{\forall n \in \mathbb{Z}, \quad n^2 \neq 2.}$$

Do not write merely

$$\forall n, \quad n^2 = 2.$$

Both the quantifier and the property must change.

TMUA Example 3

Negate:

$$\boxed{\forall x \in \mathbb{R}, \quad x^2 \geq x.}$$

The negation is

$$\boxed{\exists x \in \mathbb{R} \text{ such that } x^2 < x.}$$

For example,

$$x = \frac{1}{2}$$

satisfies

$$\frac{1}{4} < \frac{1}{2}.$$

Thus the original statement is false.

TMUA Example 4

Consider:

$$\forall n \in \mathbb{Z}, \quad n \text{ even} \Rightarrow n^2 \text{ even.}$$

What would a counterexample need to satisfy?

It would need

$$n \text{ even}$$

and

$$n^2 \text{ not even.}$$

That is,

$$n \text{ even and } n^2 \text{ odd.}$$

No such integer exists.

TMUA Example 5

Which statement is stronger?

$$(I) \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad y > x$$

or

$$(II) \quad \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, \quad y > x?$$

Statement II requires one fixed y to work for every x .

Thus it imposes a much stronger requirement.

Over $\mathbb{R}$,

I is true while II is false.

Problem 1

Consider:

$$\forall x \in \mathbb{R}, \quad x^2 + 1 > 0.$$

How would you prove it?
Since

$$x^2 \geq 0$$

for every real x ,

$$x^2 + 1 \geq 1 > 0.$$

Therefore

$$\boxed{\forall x \in \mathbb{R}, \quad x^2 + 1 > 0.}$$

Problem 2

Consider:

$$\exists x \in \mathbb{R}, \quad x^3 - x = 0.$$

To prove existence, one witness is enough.
Take

$$x = 0.$$

Then

$$0^3 - 0 = 0.$$

Therefore the existential statement is true.

Problem 3

Claim:

$$\forall x \in \mathbb{R}, \quad x^2 \geq x.$$

Search for a counterexample.
We need

$$x^2 < x.$$

Rearrange:

$$x(x - 1) < 0.$$

This occurs when

$$0 < x < 1.$$

Therefore any value in this interval is a counterexample.
For example,

Problem 3 (continued)

$$x = \frac{1}{2}.$$

Problem 4

Negate:

$$\boxed{\forall x \in \mathbb{R}, \quad x > 0 \Rightarrow f(x) > 0.}$$

First change

$$\forall \rightarrow \exists.$$

Then negate the implication.
Since

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q,$$

the negation is

$$\boxed{\exists x \in \mathbb{R} : x > 0 \text{ and } f(x) \leq 0.}$$

Problem 5

Negate:

$$\boxed{\exists x \in \mathbb{R}, \quad x > 0 \text{ and } x^2 < 1.}$$

We have

$$\neg \exists x \implies \forall x \neg.$$

So:

$$\forall x \in \mathbb{R}, \quad \neg(x > 0 \wedge x^2 < 1).$$

By De Morgan's law:

$$\boxed{\forall x \in \mathbb{R}, \quad x \leq 0 \text{ or } x^2 \geq 1.}$$

“Not Every” Does Not Mean “None”

Suppose:

Not every student solved the problem.

This means:

At least one student did not solve it.

It does not mean:

No student solved it.

The same distinction is fundamental in mathematical logic.

“Not All”

If

$$\neg(\forall x, P(x)),$$

then

$$\exists x, \neg P(x).$$

So:

not all

means

at least one exception.

It does not mean that the property fails everywhere.

“There Is No”

The statement

There is no x satisfying $P(x)$

means

$$\neg \exists x, P(x).$$

Equivalently:

$$\forall x, \neg P(x).$$

This translation appears frequently in proof arguments.

Read the Scope Carefully

Consider

$$\forall x, P(x) \rightarrow Q(x).$$

The quantifier applies to the entire implication.
Do not read it as though it merely says

$$\forall x, P(x)$$

followed by a separate statement $Q(x)$.
Logical scope determines meaning.

Variables Can Depend on Earlier Variables

In

$$\forall x \exists y, \quad P(x, y),$$

the value of y may depend on the chosen x .

For example:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad y = x^2.$$

Given x ,
choose

$$y = x^2.$$

There is no requirement that the same y work for every x .

Quantifier Order Encodes Dependence

Compare:

$$\forall x \exists y$$

with

$$\exists y \forall x.$$

In the first,

$$y = y(x)$$

may depend on x .

In the second,

$$y$$

must be chosen first and remain fixed.

Quantifier Order Encodes Dependence (continued)

Mathematical Maturity

Quantifier order is not notation decoration. It describes the dependency structure of a mathematical claim.

Ask the Right Question

For a universal claim, ask:

Can I find an exception?

For an existential claim, ask:

Can I construct one example?

For nested quantifiers, ask:

Which variable may depend on which?

These questions often reveal the logical structure immediately.

Challenge 1

Negate:

$$\forall x \in \mathbb{R}, \quad x + 1 > 0.$$

Challenge 2

Negate:

$$\exists n \in \mathbb{Z}, \quad n^2 = 7.$$

Challenge 3

Negate:

$$\forall x \in \mathbb{R}, \quad x > 2 \Rightarrow x^2 > 4.$$

Challenge 4

Which statement is the negation of

$$\forall x \exists y, \quad P(x, y)?$$

Challenge 5

Is this true?

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad xy = 1.$$

Challenge 6

Is this true?

$$\forall x \in \mathbb{R} \setminus \{0\}, \exists y \in \mathbb{R}, \quad xy = 1.$$

Challenge 7

Consider:

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x + y = y.$$

Does such an x exist?

Challenge 8

Consider:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, \quad x + y = 0.$$

This is also true.

Given y ,
choose

$$\boxed{x = -y}.$$

But notice:
the reason is different.
Here x may depend on y .

Common Error 1: Negating Every as None

Wrong:

$$\neg(\forall x, P(x)) \equiv \forall x, \neg P(x).$$

Correct:

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x).$$

“Not everyone” means at least one exception.

Common Error 2: Forgetting to Negate the Predicate

Wrong:

$$\neg(\exists x, P(x)) \equiv \forall x, P(x).$$

Correct:

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x).$$

Both the quantifier and the property change.

Common Error 3: Ignoring the Domain

The statement

$$\exists x, \quad x^2 = 2$$

has different truth values over

$$\mathbb{R}$$

and

$$\mathbb{Z}.$$

Always ask:

$x \in$ what set?

Common Error 4: Proving by Examples

Testing

$$n = 1, 2, 3, \dots, 100$$

does not normally prove

$$\forall n \in \mathbb{N}, \quad P(n).$$

It verifies only the tested cases.

A universal proof must cover every relevant n .

Common Error 5: Treating Quantifiers as Commutative

In general,

$$\forall x \exists y \neq \exists y \forall x.$$

Changing quantifier order can completely change the statement.
Unlike ordinary multiplication,
quantifiers cannot simply be rearranged.

Common Error 6: Weak Counterexamples

To disprove

$$\forall x, \quad P(x) \rightarrow Q(x),$$

you need

$$P(x) \text{ true}$$

and

$$Q(x) \text{ false.}$$

A value for which $P(x)$ is already false is not a counterexample.

The Two Fundamental Quantifiers

Universal:

$$\forall x, P(x)$$

means

$$P(x) \text{ holds for every } x.$$

Existential:

$$\exists x, P(x)$$

means

$$P(x) \text{ holds for at least one } x.$$

The Fundamental Negations

Memorise through understanding:

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$$

and

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x).$$

In words:

$$\text{not all} = \text{at least one not}$$

and

$$\text{none} = \text{all not}.$$

Proof and Disproof Map

Claim	Typical Logical Task
$\forall x, P(x)$	prove for arbitrary x
$\neg(\forall x, P(x))$	find a counterexample
$\exists x, P(x)$	find or establish a witness
$\neg(\exists x, P(x))$	show every x fails

Quantifier Order

Remember the difference:

$$\boxed{\forall x \exists y}$$

allows y to depend on x ,
while

$$\boxed{\exists y \forall x}$$

requires one fixed y to work for every x .
This distinction becomes increasingly important in advanced mathematics.

Self-Check

Before moving on, make sure you can:

- ➊ Explain the meaning of $\forall$.
- ➋ Explain the meaning of $\exists$.
- ➌ Identify the domain of a quantified statement.
- ➍ Disprove a universal claim with a counterexample.
- ➎ Prove a simple existential claim with a witness.
- ➏ Negate a universal statement.
- ➐ Negate an existential statement.
- ➑ Negate a quantified implication.
- ➒ Distinguish existence from unique existence.
- ➓ Explain why quantifier order matters.

The TMUA Habit

When reading a quantified statement, ask:

What is the domain?

Is the claim universal or existential?

What would its exact negation be?

What would a counterexample look like?

Which variables may depend on which?

These questions turn complicated language into precise mathematics.

Coming in Part 6

Proof: Direct Proof and Mathematical Arguments

We will move from understanding statements to

proving them.

We will study:

direct proof

proof from definitions

Coming in Part 6 (continued)

chains of implications

and

how to construct a valid mathematical argument.

The emphasis will shift from

“What does the statement mean?”

to

“Why must it be true?”

Read Every Quantifier. Negate Precisely. One Counterexample Can Change Everything.

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 6: Direct Proof and Constructing Mathematical Arguments
From Mathematical Statements to Valid Reasoning

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From Understanding to Proving

In the previous parts, we learned to understand statements such as

$$P \rightarrow Q,$$

$$P \leftrightarrow Q,$$

$$\forall x P(x),$$

and

$$\exists x P(x).$$

But understanding what a statement means is only the beginning.
The next question is:

Why must the statement be true?

That is the beginning of proof.

What Is a Mathematical Proof?

A mathematical proof is a sequence of logically justified steps that begins with accepted information and reaches a required conclusion.

Schematically:

Assumptions $\longrightarrow$ Valid deductions $\longrightarrow$ Conclusion.

Every arrow matters.

Proof Is More Than Calculation

A calculation may tell us that

$$(2n + 1)^2 = 4n^2 + 4n + 1.$$

But a proof explains why this establishes something.
For example,

$$(2n + 1)^2 = 2(2n^2 + 2n) + 1.$$

Since

$$2n^2 + 2n \in \mathbb{Z},$$

the expression has the form

$$2k + 1.$$

Therefore it is odd.

Proof Is More Than Calculation (continued)

Key Idea

A proof connects algebraic work to mathematical meaning.

Part 6 Roadmap

We will study:

- ① What constitutes a proof
- ② Hypotheses and conclusions
- ③ Direct proof
- ④ Using definitions correctly
- ⑤ Chains of implications
- ⑥ Proofs involving parity
- ⑦ Proofs involving divisibility
- ⑧ Algebraic proofs
- ⑨ Inequality arguments
- ⑩ Valid and invalid deductions
- ⑪ Finding the missing step
- ⑫ TMUA-style proof reasoning

Hypothesis and Conclusion

Suppose we want to prove

$$\boxed{P \rightarrow Q}.$$

Here:

$$P$$

is the hypothesis or assumption,
and

$$Q$$

is the conclusion.

A direct proof begins by assuming P and logically deriving Q .

The Basic Direct-Proof Structure

To prove

$$P \rightarrow Q,$$

the standard structure is:

Assume P .

Then use definitions, algebra, known results, or previous deductions.
Eventually obtain:

$$Q.$$

Therefore:

$$P \rightarrow Q.$$

The Logical Flow

A proof might have the form

$$P \rightarrow A,$$

$$A \rightarrow B,$$

$$B \rightarrow C,$$

$$C \rightarrow Q.$$

Therefore:

$$P \rightarrow A \rightarrow B \rightarrow C \rightarrow Q.$$

By transitivity of implication,

$$P \rightarrow Q.$$

Every Step Needs a Reason

A valid proof cannot simply say:

P

therefore

Q .

unless the implication is already justified.

Typical reasons include:

- a definition,
- an assumption,
- algebraic equivalence,
- an established theorem,
- a previously proved result,
- a valid logical inference.

Definitions Are Operational

Definitions are not merely facts to memorise.

They tell us what mathematical objects **are**.

For example:

$$n \text{ is even} \iff n = 2k \text{ for some } k \in \mathbb{Z}.$$

Therefore, if a proof says

n is even,

we can immediately write

$$n = 2k$$

for some integer k .

Definition of Odd

An integer n is odd precisely when

$$n = 2k + 1$$

for some

$$k \in \mathbb{Z}.$$

Thus the hypothesis

$$n \text{ is odd}$$

can be converted into an algebraic form.

This conversion often makes the proof possible.

Definition of Divisibility

For integers a, b ,

$$a \mid b$$

means that there exists some integer k such that

$$b = ak.$$

So if

$$3 \mid n,$$

we may write

$$n = 3k$$

for some integer k .

Translate Before You Manipulate

Suppose we want to prove:

$$n \text{ even} \Rightarrow n^2 \text{ even.}$$

The useful first step is not to try random values.
It is to translate the definition:

$$n \text{ even} \Rightarrow n = 2k.$$

Then:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Now the conclusion follows from the definition of evenness.

Example 1: Even Plus Even

Prove:

The sum of two even integers is even.

Let a and b be even integers.
Then

$$a = 2m, \quad b = 2n$$

for some $m, n \in \mathbb{Z}$.
Therefore

$$a + b = 2m + 2n = 2(m + n).$$

Since $m + n \in \mathbb{Z}$,

$a + b$ is even.

What Made the Proof Work?

The argument was:

$$a, b \text{ even}$$

$$\Downarrow$$

$$a = 2m, \quad b = 2n$$

$$\Downarrow$$

$$a + b = 2(m + n)$$

$$\Downarrow$$

What Made the Proof Work? (continued)

$$a + b \text{ even.}$$

The proof begins and ends with definitions.

Example 2: Odd Plus Odd

Let

$$a = 2m + 1, \quad b = 2n + 1.$$

Then

$$a + b = 2m + 1 + 2n + 1$$

$$= 2m + 2n + 2$$

$$= 2(m + n + 1).$$

Since $m + n + 1 \in \mathbb{Z}$,

$$\boxed{a + b \text{ is even.}}$$

Example 3: Odd Times Odd

Suppose a and b are odd.

Then

$$a = 2m + 1, \quad b = 2n + 1.$$

Hence

$$ab = (2m + 1)(2n + 1)$$

$$= 4mn + 2m + 2n + 1$$

$$= 2(2mn + m + n) + 1.$$

Therefore

$$\boxed{ab \text{ is odd.}}$$

A More Structural Proof

Suppose n is odd.
Then

$$n = 2k + 1.$$

Therefore

$$n^2 = 4k^2 + 4k + 1$$

$$= 4k(k + 1) + 1.$$

Hence

$$n^2 \equiv 1 \pmod{4}.$$

This proves something stronger than merely saying that n^2 is odd.

A More Structural Proof (continued)

Mathematical Maturity

Sometimes the structure of a proof reveals a stronger theorem than the one originally requested.

Example 4: Divisibility Chain

Prove:

$$6 \mid n \Rightarrow 3 \mid n.$$

Assume

$$6 \mid n.$$

Then, by definition,

$$n = 6k$$

for some integer k .

But

$$n = 3(2k).$$

Since $2k \in \mathbb{Z}$,

Example 4: Divisibility Chain (continued)

$$3 \mid n.$$

Example 5: Product Divisibility

Suppose

$$a \mid b.$$

Prove that

$$a \mid bc$$

for every integer c .

Since

$$a \mid b,$$

there exists $k \in \mathbb{Z}$ such that

$$b = ak.$$

Then

Example 5: Product Divisibility (continued)

$$bc = (ak)c = a(kc).$$

Since $kc \in \mathbb{Z}$,

$$\boxed{a \mid bc}.$$

Example 6: Combining Divisibility

Suppose

$$a \mid b$$

and

$$a \mid c.$$

Then there exist integers m, n such that

$$b = am, \quad c = an.$$

Therefore

$$b + c = a(m + n).$$

Since $m + n \in \mathbb{Z}$,

$$\boxed{a \mid (b + c)}.$$

A Useful Generalisation

From

$$a \mid b$$

and

$$a \mid c,$$

we can prove that for any integers r, s ,

$$\boxed{a \mid (rb + sc)}.$$

Indeed,

$$rb + sc = r(am) + s(an)$$

$$= a(rm + sn).$$

This idea of integer linear combinations is extremely useful.

Example 7: Difference of Squares

Prove that for real x, y ,

$$x = y \Rightarrow x^2 - y^2 = 0.$$

Assume

$$x = y.$$

Then

$$x - y = 0.$$

Using

$$x^2 - y^2 = (x - y)(x + y),$$

we obtain

$$x^2 - y^2 = 0.$$

Therefore the implication is true.

Example 8: A Positive Expression

Prove that for every real x ,

$$x^2 - 4x + 5 > 0.$$

Complete the square:

$$x^2 - 4x + 5 = (x - 2)^2 + 1.$$

For every real x ,

$$(x - 2)^2 \geq 0.$$

Therefore

$$(x - 2)^2 + 1 \geq 1 > 0.$$

Hence

$$x^2 - 4x + 5 > 0$$

Example 8: A Positive Expression (continued)

for every real x .

The Structural Step

The important step was not expanding.
It was rewriting

$$x^2 - 4x + 5$$

as

$$(x - 2)^2 + 1.$$

This exposes a known fact:

$$(x - 2)^2 \geq 0.$$

TMUA Thinking

In reasoning problems, the best algebraic form is often the form that exposes the logical conclusion.

Example 9

Prove that for $x > 0$,

$$x + \frac{1}{x} \geq 2.$$

Since

$$(x - 1)^2 \geq 0,$$

we have

$$x^2 - 2x + 1 \geq 0.$$

Therefore

$$x^2 + 1 \geq 2x.$$

Since $x > 0$, dividing by x preserves the inequality:

Example 9 (continued)

$$x + \frac{1}{x} \geq 2.$$

Why the Condition $x > 0$ Matters

We divided by x .

If

$$x > 0,$$

the direction of the inequality is unchanged.

If

$$x < 0,$$

the direction would reverse.

If

$$x = 0,$$

division is impossible.

Therefore the assumption

$x > 0$

Why the Condition $x > 0$ Matters (continued)

is not decorative.

It is logically necessary for this argument.

Track Assumptions

Whenever you use an assumption, ask:

Where exactly did I need it?

For the previous proof,

$$x > 0$$

was needed when dividing by x .
This habit helps distinguish:

essential assumptions

from

unnecessary assumptions.

Forward Reasoning

Suppose we want to prove

$$P \rightarrow Q.$$

Forward reasoning asks:

What follows from P ?

We begin with the hypothesis and derive consequences:

$$P \rightarrow A \rightarrow B \rightarrow \dots$$

hoping eventually to reach Q .

Backward Thinking

Another useful question is:

What would be enough to establish Q ?

Suppose we want to prove

$$x^2 - 4x + 5 > 0.$$

We might think backward:

$$\text{positive} \Leftarrow \text{square} + 1$$

which suggests completing the square.

Backward thinking helps discover a proof.

Discovery vs Presentation

During problem solving, you may think:

forward from assumptions

and

backward from the goal.

These ideas may meet in the middle.

But the final proof should normally be presented as a clear logical argument from assumptions to conclusion.

Mathematical Maturity

Proof discovery can be messy. Proof presentation should be clear.

Working Backwards Can Be Dangerous

Suppose the goal is

A .

You manipulate it and obtain

B .

Then prove B .

Have you necessarily proved A ?

Only if your transformations were reversible, or if you know

$$\boxed{B \rightarrow A}.$$

Otherwise the reasoning may be invalid.

Example of Reversible Reasoning

For real x ,

$$x^2 - 6x + 9 = 0$$

is equivalent to

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x = 3.$$

Thus:

$$x^2 - 6x + 9 = 0 \iff x = 3.$$

Every transformation is reversible.

Example of a Non-Reversible Step

Suppose

$$x = 2.$$

Squaring gives

$$x^2 = 4.$$

But the reverse implication fails:

$$x^2 = 4 \not\Rightarrow x = 2,$$

because

$$x = -2$$

also satisfies $x^2 = 4$.

Therefore squaring is not automatically an equivalence.

Implication vs Equivalence

When solving or proving, distinguish:

$$\Rightarrow$$

from

$$\Leftrightarrow.$$

Use

$$A \Rightarrow B$$

when A guarantees B .

Use

$$A \Leftrightarrow B$$

only when both directions are valid.

This prevents many subtle errors.

Modus Ponens

Suppose:

$$P \rightarrow Q$$

and

$$P.$$

Then we may conclude:

$$\boxed{Q}.$$

This argument form is called **modus ponens**.
Schematically:

$$\boxed{P, \quad P \rightarrow Q \quad \therefore \quad Q.}$$

Example of Modus Ponens

Suppose:

$$n > 5 \Rightarrow n^2 > 25$$

for the relevant real n ,
and we know

$$n = 7.$$

Then

$$n > 5.$$

Therefore

$$\boxed{n^2 > 25}.$$

The conclusion follows through a valid implication.

Modus Tollens

Suppose:

$$P \rightarrow Q$$

and

$$\neg Q.$$

Then the contrapositive gives

$$\boxed{\neg P}.$$

Schematically:

$$\boxed{P \rightarrow Q, \neg Q \therefore \neg P.}$$

This is called **modus tollens**.

Example of Modus Tollens

Suppose:

$$4 \mid n \Rightarrow n \text{ even.}$$

Now suppose

$$n \text{ is odd.}$$

Then n is not even.

Therefore:

$$\boxed{4 \nmid n}.$$

This is an application of the contrapositive.

The Converse Error

Suppose:

$$P \rightarrow Q$$

and

$$Q.$$

Can we conclude P ?

No.

This would be:

$$P \rightarrow Q, \quad Q \quad \therefore \quad P.$$

This is generally invalid.

Example of the Converse Error

We know:

$$4 \mid n \Rightarrow n \text{ even.}$$

Suppose n is even.

Can we conclude

$$4 \mid n?$$

No.

Take

$$n = 6.$$

Thus the argument

$$Q, \quad P \rightarrow Q \quad \therefore \quad P$$

is invalid.

Denying the Antecedent

Suppose:

$$P \rightarrow Q$$

and

$$\neg P.$$

Can we conclude

$$\neg Q?$$

No.

The argument

$$P \rightarrow Q, \neg P \therefore \neg Q$$

is generally invalid.

Example

We know:

$$4 \mid n \Rightarrow n \text{ even.}$$

Suppose

$$4 \nmid n.$$

Can we conclude n is odd?

No.

For example,

$$n = 6$$

is not divisible by 4,
but it is even.

Combining Implications

Suppose:

$$P \rightarrow Q$$

and

$$Q \rightarrow R.$$

Then:

$$\boxed{P \rightarrow R.}$$

This is the logical foundation of multi-step mathematical arguments.

Example

Suppose n is divisible by 12.

Then:

$$12 \mid n \Rightarrow 6 \mid n$$

and

$$6 \mid n \Rightarrow 3 \mid n.$$

Therefore:

$$\boxed{12 \mid n \Rightarrow 3 \mid n.}$$

The intermediate statement connects the hypothesis to the conclusion.

Finding a Missing Step

Suppose:

$$P \rightarrow Q,$$

$$?,$$

$$R \rightarrow S.$$

To establish

$$P \rightarrow S,$$

a natural missing link is

$$\boxed{Q \rightarrow R.}$$

Then:

Finding a Missing Step (continued)

$$P \rightarrow Q \rightarrow R \rightarrow S.$$

TMUA Paper 2 can test whether you can identify such logical connections efficiently.

Universal Implications

Many mathematical theorems have the form

$$\forall x \in D, \quad P(x) \rightarrow Q(x).$$

To prove such a statement directly:

- ❶ Choose an arbitrary $x \in D$.
- ❷ Assume $P(x)$.
- ❸ Deduce $Q(x)$.
- ❹ Since x was arbitrary, conclude the universal claim.

Example

Prove:

$$\forall n \in \mathbb{Z}, \quad 6 \mid n \Rightarrow 2 \mid n.$$

Let n be an arbitrary integer satisfying

$$6 \mid n.$$

Then

$$n = 6k$$

for some $k \in \mathbb{Z}$.

Hence

$$n = 2(3k).$$

Therefore

Example (continued)

$$2 \mid n.$$

Since n was arbitrary, the result holds universally.

Direct Proof of Existence

Not every direct proof proves an implication.
Suppose we want to establish

$$\boxed{\exists x, P(x)}.$$

A direct method is to construct a suitable x .
Then verify that it satisfies $P(x)$.

Example

Prove:

$$\forall x \in \mathbb{R}, \quad \exists y \in \mathbb{R} \text{ such that } y > x.$$

Let $x \in \mathbb{R}$ be arbitrary.

Choose

$$y = x + 1.$$

Then

$$y - x = 1 > 0,$$

so

$$y > x.$$

Thus such a y exists for every real x .

The Witness May Depend on the Input

Notice that

$$y = x + 1$$

depends on x .

This is permitted because the statement is

$$\forall x \exists y.$$

For each x ,

we may construct a different y .

This connects directly to the quantifier ideas from Part 5.

TMUA Example 1

Suppose:

$$P \rightarrow Q,$$

$$Q \rightarrow R,$$

and P is true.

Which conclusion must be true?

(A) $\neg Q$

(B) R

(C) $\neg R$

(D) $R \rightarrow P$

TMUA Example 1 (continued)

Since

$$P \rightarrow Q \rightarrow R,$$

the answer is

(B).

TMUA Example 2

Suppose:

$$P \rightarrow Q$$

and

Q is false.

What must follow?
By the contrapositive,

$$\neg Q \rightarrow \neg P.$$

Therefore

$$\boxed{\neg P}.$$

TMUA Example 3

Suppose:

$$P \rightarrow Q$$

and

Q is true.

Can we determine whether P is true?

$\boxed{\text{No.}}$

The implication allows:

$$P = T, Q = T$$

or

$$P = F, Q = T.$$

Thus Q alone does not determine P .

TMUA Example 4

A student argues:

$$n^2 \text{ is even}$$

because

$$n = 2k$$

and therefore

$$n^2 = 4k^2.$$

What assumption is missing?

The student has assumed

$n \text{ is even.}$

Without this hypothesis, writing

TMUA Example 4 (continued)

$$n = 2k$$

is unjustified.

TMUA Example 5

A student wants to prove:

$$x > 2 \Rightarrow x^2 > 4.$$

They write:

$$x^2 > 4 \Rightarrow x > 2.$$

What is wrong?

The reverse implication is false.

For example,

$$x = -3$$

satisfies

$$x^2 > 4$$

but not $x > 2$.

The student has reversed the required implication.

Do Not Ask Only “Is the Conclusion True?”

Suppose an argument reaches a true conclusion.

That does not automatically make the argument valid.

For example:

$$2 < 3$$

is true.

But an invalid chain of reasoning that happens to end at

$$2 < 3$$

is still an invalid argument.

Key Idea

Validity concerns how the conclusion follows, not merely whether the conclusion happens to be true.

Truth vs Validity

These are different questions:

Is the statement true?

and

Does the conclusion follow from the premises?

A logically valid argument can begin from false premises.

A true conclusion can also be reached through invalid reasoning.

Paper 2 requires you to separate these ideas.

Example

Consider:

All squares are rectangles.

This quadrilateral is a rectangle.

Therefore:

This quadrilateral is a square.

The conclusion might happen to be true for a particular quadrilateral.

But it does not follow logically from the premises.

The argument form is invalid.

Hidden Assumptions

A proof may contain a step that is valid only under an unstated condition.
For example:

$$xy = xz$$

therefore

$$y = z.$$

This is valid only if

$$\boxed{x \neq 0}.$$

If $x = 0$,
then

$$0y = 0z$$

for all y, z .

Division Requires Non-Zero Quantities

Whenever a proof divides by an expression A ,
you must know

$$\boxed{A \neq 0}.$$

For example:

$$(x - 1)(x + 2) = 0.$$

Dividing immediately by $x - 1$ may lose the solution

$$x = 1.$$

The operation changes the logical possibilities unless the divisor is known to be non-zero.

Square Roots Require Care

From

$$x^2 = 9,$$

we cannot conclude only

$$x = 3.$$

We must conclude

$$\boxed{x = \pm 3}.$$

Similarly,

$$\sqrt{x^2} = |x|,$$

not always x .

Proofs depend on precise algebraic statements.

A Stronger Statement

Suppose

$$P \rightarrow Q.$$

Then P is stronger than Q in the sense that whenever P is true, Q must also be true.

For example:

$$x > 10 \Rightarrow x > 5.$$

Thus

$$x > 10$$

is a stronger condition than

$$x > 5.$$

Why This Helps in Proof

Suppose the goal is to prove Q .

If we can prove some stronger statement P such that

$$P \rightarrow Q,$$

then Q follows automatically.

For example, proving

$$n^2 \equiv 1 \pmod{8}$$

for a suitable class of odd integers is stronger than merely proving that n^2 is odd.

But Stronger Can Be Harder

A stronger claim contains more information.

Therefore it may be harder to prove.

A good problem solver asks:

What is the weakest statement sufficient for my goal?

This can prevent unnecessary work.

Challenge 1

Prove that the sum of an even integer and an odd integer is odd.

Challenge 2

Suppose

$$5 \mid a$$

and

$$5 \mid b.$$

Prove

$$5 \mid (3a - 2b).$$

Challenge 2 (continued)

Therefore

$$5 \mid (3a - 2b).$$

Challenge 3

Prove that for every real x ,

$$x^2 + 2x + 2 > 0.$$

Challenge 4

Suppose

$$P \rightarrow Q,$$

$$R \rightarrow P,$$

and

$$Q \rightarrow S.$$

What follows from R ?

Challenge 5

Suppose

$$P \rightarrow Q$$

and

$$\neg P.$$

Can you conclude $\neg Q$?

Challenge 6

A student writes:

$$x^2 = 4 \Rightarrow x = 2.$$

Find the error.

Challenge 7

Prove:

$$\forall n \in \mathbb{Z}, \quad n \text{ odd} \Rightarrow n^2 \text{ odd}.$$

Challenge 8

For every real x , prove that there exists a real y such that

$$y^2 \geq x.$$

Common Error 1: Starting With the Conclusion

To prove

$$P \rightarrow Q,$$

you should not simply assume Q .

The direct-proof structure is:

Assume P	$\longrightarrow$	derive Q .
------------	-------------------	--------------

Starting from the conclusion may be useful for exploration, but it is not automatically a proof.

Common Error 2: Testing Examples

To prove:

$$\forall n \in \mathbb{Z}, \quad n \text{ even} \Rightarrow n^2 \text{ even},$$

checking

$$n = 2, 4, 6, 8$$

is not sufficient.

A proof must handle an arbitrary even integer:

$$\boxed{n = 2k}.$$

Common Error 3: Using the Converse

From

$$P \rightarrow Q,$$

do not automatically use

$$Q \rightarrow P.$$

The converse requires separate justification.

This is one of the most important logical traps in Paper 2.

Common Error 4: Dividing by Zero

From

$$xy = xz,$$

do not divide by x unless

$$x \neq 0.$$

Any algebraic operation used in a proof must be valid under the current assumptions.

Common Error 5: Losing Cases

From

$$x^2 = 9,$$

writing only

$$x = 3$$

loses the case

$$x = -3.$$

Similarly, dividing by a factor may remove a possible zero case.

Always ask:

Have I excluded a legitimate possibility?

Common Error 6: Unjustified Algebra

A line such as

$$A = B$$

must follow from what came before.

Do not let familiar-looking algebra hide an invalid operation.

Check:

Was anything divided by zero?

Was an implication reversed?

Were extra solutions introduced?

Were solutions lost?

A Practical Direct-Proof Framework

When asked to prove a statement directly:

1. Identify the hypothesis.

2. Identify the conclusion.

3. Expand useful definitions.

4. Ask what follows from the hypothesis.

5. Ask what would establish the conclusion.

6. Build a valid bridge between them.

The Proof Bridge

Think of a proof as constructing a bridge:

What I Know



Definitions



Intermediate Results



The Proof Bridge (continued)

Known Theorems / Algebra



What I Need to Show.

When You Are Stuck

Ask:

What exactly am I assuming?

What exactly must I prove?

Which definitions can I expand?

What does the hypothesis immediately imply?

What condition would be sufficient for the goal?

Often the problem becomes easier once its logical structure is made explicit.

A Proof Is a Dependency Structure

A mature proof is not merely a sequence of equations.
It records dependencies:

Because A , we know B .

Because B , we know C .

Therefore

$$A \rightarrow B \rightarrow C.$$

Understanding these dependencies is central to Paper 2.

Know Why Each Assumption Exists

Suppose a theorem states:

$$\text{If } x > 0, \text{ then } x + \frac{1}{x} \geq 2.$$

Do not merely remember the theorem.
Ask:

Why $x > 0$?

What breaks if $x = 0$?

What changes if $x < 0$?

These questions reveal the architecture of the theorem.

How, Why, and What If?

For every mathematical result, develop four levels:

What does it say?

How do I use it?

Why does it work?

What if the assumptions change?

This moves learning from procedural familiarity toward mathematical maturity.

Proof Is Controlled Necessity

A proof does not say:

The conclusion seems plausible.

It says:

Given these assumptions, the conclusion cannot fail.

That distinction separates mathematical proof from evidence, experimentation, and intuition.

Direct Proof

To prove

$$P \rightarrow Q$$

directly:

Assume P

then derive consequences through valid reasoning until

$$Q$$

is obtained.

Therefore:

$$P \rightarrow Q.$$

The Core Logical Tools

You should now recognise:

$$P, P \rightarrow Q \therefore Q$$

as valid,
and

$$P \rightarrow Q, \neg Q \therefore \neg P$$

as valid.
But:

$$P \rightarrow Q, Q \therefore P$$

is generally invalid.
And:

$$P \rightarrow Q, \neg P \therefore \neg Q$$

is generally invalid.

Proof Checklist

Before accepting an argument, ask:

- ❶ What are the assumptions?
- ❷ What is the required conclusion?
- ❸ Is every definition used correctly?
- ❹ Does every step follow?
- ❺ Has an implication been reversed?
- ❻ Has a non-reversible operation been treated as reversible?
- ❼ Has division by zero occurred?
- ❽ Have any cases been lost?
- ❾ Have any extra cases been introduced?
- ❿ Does the final line actually establish the required result?

Self-Check

Before moving on, make sure you can:

- ① Identify hypothesis and conclusion.
- ② Construct a basic direct proof.
- ③ Use definitions of even, odd, and divisibility.
- ④ Build a chain of implications.
- ⑤ Distinguish implication from equivalence.
- ⑥ Recognise modus ponens.
- ⑦ Recognise modus tollens.
- ⑧ Detect the converse error.
- ⑨ Detect denying the antecedent.
- ⑩ Identify hidden assumptions in algebraic arguments.

The TMUA Habit

When examining a mathematical argument, do not ask only:

Is the final statement true?

Ask:

Does each step follow?

Which assumption justifies it?

Is the implication going in the correct direction?

Could there be a hidden exception?

That is the mindset of mathematical reasoning.

Proof by Contrapositive and Contradiction

Sometimes

$$P \rightarrow Q$$

is difficult to prove directly.
Instead, we may prove the equivalent statement

$$\neg Q \rightarrow \neg P$$

or assume that the required conclusion fails and derive an impossibility.

We will study:

Proof by Contrapositive

Proof by Contradiction

Choosing the Better Proof Method

and

Recognising Invalid Contradictions.

Know What You Assume. Know What You Must Prove. Justify the Bridge Between Them.

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 7: Proof by Contrapositive and Contradiction
When the Direct Route Is Not the Best Route

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Not Every Proof Should Be Direct

Suppose we want to prove

$$P \rightarrow Q.$$

A direct proof begins with P and tries to reach Q .

But sometimes this route is awkward.

The conclusion may be negative, such as

$$x \neq 0,$$

n is not odd,

or

x is irrational.

In such cases, an indirect argument may expose the structure more naturally.

Two Powerful Indirect Methods

We will develop two fundamental techniques:

Proof by Contrapositive

and

Proof by Contradiction.

These methods do not weaken mathematical rigour.

They simply approach the logical structure from a different direction.

Part 7 Roadmap

We will study:

- ① Contrapositive statements
- ② Why the contrapositive is equivalent
- ③ Proof by contrapositive
- ④ Contrapositive vs converse
- ⑤ Proof by contradiction
- ⑥ What counts as a contradiction
- ⑦ Parity and divisibility examples
- ⑧ Irrationality arguments
- ⑨ Impossibility arguments
- ⑩ Negating the conclusion correctly
- ⑪ Choosing between proof methods
- ⑫ Detecting invalid indirect proofs

Recall an Implication

Suppose

$$P \rightarrow Q.$$

Its contrapositive is

$$\neg Q \rightarrow \neg P.$$

These statements are logically equivalent:

$$(P \rightarrow Q) \iff (\neg Q \rightarrow \neg P).$$

Therefore proving the contrapositive proves the original implication.

Why Are They Equivalent?

An implication

$$P \rightarrow Q$$

is false only when

$$P = T, \quad Q = F.$$

Now consider

$$\neg Q \rightarrow \neg P.$$

It is false only when

$$\neg Q = T, \quad \neg P = F,$$

which means

$$Q = F, \quad P = T.$$

Exactly the same case makes both statements false.

Truth Table

P	Q	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The final two columns agree in every row.

Hence

$$P \rightarrow Q \equiv \neg Q \rightarrow \neg P.$$

Do Not Confuse Four Statements

Starting with

$$P \rightarrow Q,$$

we have:

Statement	Form
Original	$P \rightarrow Q$
Converse	$Q \rightarrow P$
Inverse	$\neg P \rightarrow \neg Q$
Contrapositive	$\neg Q \rightarrow \neg P$

Only the contrapositive is automatically equivalent to the original implication.

Example

Original:

$$4 \mid n \Rightarrow n \text{ is even.}$$

Contrapositive:

$$n \text{ is odd} \Rightarrow 4 \nmid n.$$

Converse:

$$n \text{ even} \Rightarrow 4 \mid n.$$

The converse is false.
For example,

$$n = 6.$$

The Method

To prove

$$P \rightarrow Q$$

by contrapositive:

$$\boxed{\text{Assume } \neg Q}$$

and prove

$$\boxed{\neg P}.$$

Since

$$\neg Q \rightarrow \neg P$$

is equivalent to

$$P \rightarrow Q,$$

the original theorem follows.

When Is Contrapositive Useful?

Contrapositive proof is especially useful when:

- the conclusion Q is difficult to establish directly;
- the negation $\neg Q$ has a useful algebraic form;
- the hypothesis involves divisibility or parity;
- the conclusion says something is non-zero;
- the conclusion says an object does not have a property.

TMUA Thinking

Do not choose a proof method mechanically. Compare the information contained in P with the information contained in $\neg Q$.

Example 1

Prove:

$$n^2 \text{ even} \Rightarrow n \text{ even}$$

for integers n .

A direct proof from

$$n^2 = 2k$$

to $n = 2m$ is not especially natural.

Consider the contrapositive instead.

Construct the Contrapositive

Original:

$$n^2 \text{ even} \Rightarrow n \text{ even.}$$

Let

$$P : n^2 \text{ even}$$

and

$$Q : n \text{ even.}$$

Then

$$\neg Q : n \text{ odd}$$

and

$$\neg P : n^2 \text{ odd.}$$

Construct the Contrapositive (continued)

So the contrapositive is

$$n \text{ odd} \Rightarrow n^2 \text{ odd.}$$

Complete the Proof

Assume n is odd.

Then

$$n = 2k + 1$$

for some $k \in \mathbb{Z}$.

Hence

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1.$$

Therefore n^2 is odd.

Hence, by contrapositive,

Complete the Proof (continued)

$$n^2 \text{ even} \Rightarrow n \text{ even.}$$

Why Contrapositive Was Natural

The original hypothesis

$$n^2 \text{ even}$$

contains information about a square.

Extracting information about n directly can be awkward.

But assuming

$$n \text{ odd}$$

immediately gives

$$n = 2k + 1.$$

The definition creates a useful algebraic representation.

Key Idea

Choose the direction that gives you the most usable starting information.

Example 2

Prove:

$$n^2 \text{ odd} \Rightarrow n \text{ odd.}$$

Its contrapositive is:

$$n \text{ even} \Rightarrow n^2 \text{ even.}$$

Assume

$$n = 2k.$$

Then

$$n^2 = 4k^2 = 2(2k^2),$$

which is even.

Therefore the original implication follows.

Example 3

Prove:

$$3 \nmid n^2 \Rightarrow 3 \nmid n.$$

The contrapositive is

$$3 \mid n \Rightarrow 3 \mid n^2.$$

Assume

$$n = 3k.$$

Then

$$n^2 = 9k^2 = 3(3k^2).$$

Therefore

$$3 \mid n^2.$$

Example 3 (continued)

Hence the original implication is true.

Example 4

Prove:

$$5 \nmid ab \Rightarrow 5 \nmid a \text{ and } 5 \nmid b.$$

The conclusion is a conjunction.
Its negation is therefore important.
If

$$Q : 5 \nmid a \wedge 5 \nmid b,$$

then

$$\neg Q : 5 \mid a \vee 5 \mid b.$$

So the contrapositive becomes

$$(5 \mid a \vee 5 \mid b) \Rightarrow 5 \mid ab.$$

Complete the Argument

Suppose

$$5 \mid a$$

or

$$5 \mid b.$$

If

$$a = 5k,$$

then

$$ab = 5(kb),$$

so $5 \mid ab$.

Similarly, if

Complete the Argument (continued)

$$b = 5m,$$

then

$$ab = 5(am).$$

Therefore

$$5 \mid ab.$$

Hence the original implication follows by contrapositive.

Example 5: Non-Zero Product

Prove:

$$xy \neq 0 \Rightarrow x \neq 0 \text{ and } y \neq 0.$$

Consider the contrapositive.
The negation of

$$x \neq 0 \wedge y \neq 0$$

is

$$x = 0 \vee y = 0.$$

Thus we need to prove:

$$x = 0 \vee y = 0 \Rightarrow xy = 0.$$

This is immediate.
Therefore the original implication holds.

Example 6

Prove:

$$x^2 + y^2 = 0 \Rightarrow x = 0 \text{ and } y = 0$$

for real x, y .

This can actually be proved directly:

$$x^2 \geq 0, \quad y^2 \geq 0.$$

If

$$x^2 + y^2 = 0,$$

neither non-negative term can be positive.
Therefore

$$x^2 = y^2 = 0,$$

and hence

Example 6 (continued)

$$x = y = 0.$$

Not Every Problem Needs an Indirect Proof

The previous example illustrates an important principle:

Use indirect proof when it simplifies the reasoning.

Do not use contrapositive merely because you know the method.
A short transparent direct proof is often preferable.

Mathematical Maturity

Proof methods are tools, not rituals.

The Central Idea

Suppose we want to prove a statement S .
Proof by contradiction begins by assuming:

$$\boxed{\neg S}.$$

We then reason logically until we obtain an impossibility.
Symbolically:

$$\neg S \rightarrow \perp,$$

where

$$\perp$$

represents a contradiction.
Therefore

$$\boxed{S}$$

must be true.

What Is a Contradiction?

A contradiction occurs when our reasoning forces mutually incompatible conclusions.
For example:

$$P \wedge \neg P,$$

or

$$x > 3 \quad \text{and} \quad x \leq 3,$$

or

$$n \text{ is both even and odd,}$$

or

$$0 = 1.$$

If valid reasoning from an assumption produces such an impossibility, the assumption must be false.

The Contradiction Structure

To prove S :

Assume $\neg S$.

Then derive:

A ,

B ,

C ,

until eventually

$R \wedge \neg R$.

Therefore the assumption $\neg S$ cannot hold.

Hence

The Contradiction Structure (continued)

S .

Why Contradiction Works

The assumption

$$\neg S$$

has led, through valid reasoning, to something impossible.
Therefore

$$\neg S$$

cannot be true.
Hence

$$\neg(\neg S)$$

must be true.
Thus:

$$\boxed{S}.$$

The method depends crucially on every intermediate step being valid.

Example 7: No Largest Integer

Prove that there is no largest integer.
Suppose, for contradiction, that there is a largest integer.
Call it

$$N.$$

Then

$$N + 1$$

is also an integer.
But

$$N + 1 > N.$$

This contradicts the assumption that N is the largest integer.
Therefore:

$$\boxed{\text{There is no largest integer.}}$$

Analyse the Structure

Claim:

No largest integer exists.

Negation:

A largest integer exists.

Assume such an integer N exists.

Construct

$$N + 1.$$

Then

$$N + 1 \in \mathbb{Z}$$

and

Analyse the Structure (continued)

$$N + 1 > N.$$

Contradiction.

The Construction Was the Key

Once we assumed a largest integer N ,
the natural object to construct was

$$N + 1.$$

Why?

Because it preserves membership in the integers while violating the claimed maximality.

TMUA Thinking

In contradiction arguments, ask: what object can I construct that conflicts with the assumption?

Example 8: Irrationality of $\sqrt{2}$

We want to prove:

$$\sqrt{2} \text{ is irrational.}$$

The conclusion is negative.

A natural contradiction strategy is therefore:

$$\text{Assume } \sqrt{2} \text{ is rational.}$$

Then explore what this assumption forces.

Assume the Opposite

Suppose

$$\sqrt{2} = \frac{a}{b},$$

where

$$a, b \in \mathbb{Z}, \quad b \neq 0,$$

and the fraction is in lowest terms:

$$\gcd(a, b) = 1.$$

Squaring gives

$$2 = \frac{a^2}{b^2}.$$

Therefore

Assume the Opposite (continued)

$$\boxed{a^2 = 2b^2}.$$

First Consequence

From

$$a^2 = 2b^2,$$

we see that a^2 is even.

From our earlier parity result,

$$a^2 \text{ even} \Rightarrow a \text{ even.}$$

Therefore

$$a = 2k$$

for some integer k .

Substitute this into

$$a^2 = 2b^2.$$

Second Consequence

We have

$$(2k)^2 = 2b^2.$$

Hence

$$4k^2 = 2b^2,$$

so

$$b^2 = 2k^2.$$

Thus b^2 is even.

Therefore b is even.

So:

$a \text{ is even}$

and

Second Consequence (continued)

b is even.

The Contradiction

If both a and b are even,
then both are divisible by 2.
Therefore

$$\gcd(a, b) \geq 2.$$

But we assumed that

$$\gcd(a, b) = 1.$$

Contradiction.

Hence our assumption that $\sqrt{2}$ is rational is false.
Therefore

$\sqrt{2}$ is irrational.

Why Lowest Terms Matter

Suppose we had written only

$$\sqrt{2} = \frac{a}{b}$$

without requiring the fraction to be in lowest terms.

Showing that a and b are both even would not itself be a contradiction.

Fractions can certainly have even numerator and denominator.

The contradiction arises because we assumed

$$\gcd(a, b) = 1.$$

Hidden Architecture of the Proof

The proof has the structure:

$\sqrt{2}$ rational

$\Downarrow$

$\sqrt{2} = \frac{a}{b}$ in lowest terms

$\Downarrow$

$$a^2 = 2b^2$$

$\Downarrow$

Hidden Architecture of the Proof (continued)

a even

$\Downarrow$

b even

$\Downarrow$

$\gcd(a, b) \neq 1.$

Contradiction.

Example 9

Suppose r is rational and x is irrational.
Prove that

$x + r$ is irrational.

Assume for contradiction that

$x + r$

is rational.

Then there exists a rational number q such that

$$x + r = q.$$

Hence

$$x = q - r.$$

Complete the Contradiction

Since rational numbers are closed under subtraction,

$$q - r$$

is rational.

Therefore

$$x$$

is rational.

But x was assumed irrational.

Contradiction.

Hence:

$$x + r \text{ is irrational.}$$

Contradiction Begins With the Exact Negation

If you want to prove

$$S,$$

you must assume

$$\neg S.$$

Not a vaguely related statement.

For example, to prove

$$x > 5,$$

the negation is

$$x \leq 5,$$

not merely

Contradiction Begins With the Exact Negation (continued)

$$x < 5.$$

Equality must not be forgotten.

Negating a Conjunction

Suppose the conclusion is

$$P \wedge Q.$$

Its negation is

$$\neg P \vee \neg Q.$$

Not

$$\neg P \wedge \neg Q.$$

This follows from De Morgan's law:

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

Negating an Existential Claim

Suppose we want to prove

$$\exists x, P(x).$$

Its negation is

$$\boxed{\forall x, \neg P(x)}.$$

Thus a contradiction proof of existence may begin by assuming that no possible x has the required property.

Negating a Universal Claim

Suppose we want to prove

$$\forall x, P(x).$$

Its negation is

$$\boxed{\exists x, \neg P(x)}.$$

So a contradiction proof may assume that a counterexample exists and then show that such a counterexample is impossible.

They Are Related but Different

To prove

$$P \rightarrow Q$$

by contrapositive, we prove

$$\boxed{\neg Q \rightarrow \neg P.}$$

We assume $\neg Q$ and deliberately aim for $\neg P$.

In contradiction, we assume the negation of the whole required statement and aim for any logical impossibility.

Contrapositive Has a Target

For contrapositive:

$$\neg Q$$

is the starting point,
and

$$\neg P$$

is the target.

Thus the structure is:

$$\boxed{\neg Q \longrightarrow \dots \longrightarrow \neg P.}$$

This is often highly organized.

Contradiction Has an Impossible Target

For contradiction:

$$\neg S$$

is the starting point.

The goal is to derive something impossible:

$$R \wedge \neg R$$

or an equivalent contradiction.

Thus:

$$\neg S \longrightarrow \dots \longrightarrow \perp.$$

Which Should You Use?

A useful heuristic:

Direct proof

when the hypothesis naturally generates the conclusion.

Use

Contrapositive

when $\neg Q$ gives a cleaner starting point.

Use

Contradiction

when assuming failure creates strong structural consequences or an impossible object.

Example 10

Prove that there are no real numbers x, y satisfying

$$x^2 + y^2 = -1.$$

Suppose such real numbers exist.
Since

$$x^2 \geq 0$$

and

$$y^2 \geq 0,$$

we have

$$x^2 + y^2 \geq 0.$$

But the equation says

Example 10 (continued)

$$x^2 + y^2 = -1 < 0.$$

Contradiction.

Existence and Non-Existence

A statement of the form

There is no object satisfying P

can be attacked by assuming:

There exists an object satisfying P .

Then show that the supposed object violates some necessary condition.
This is a common and powerful contradiction pattern.

A Contradiction Must Come From the Assumption

Suppose you assume $\neg S$.

Later you make an algebraic error and obtain

$$0 = 1.$$

This does not prove S .

The contradiction is meaningful only if every step leading to it is logically valid.

Common Mistake

An error in your reasoning is not a mathematical contradiction.

Do Not Assume Too Much

Suppose you want to prove S .
In contradiction, assume only

$$\neg S$$

together with the original hypotheses.

Do not introduce an extra false assumption merely because it helps produce a contradiction.
Otherwise the contradiction may come from your additional assumption rather than from $\neg S$.

Example of an Invalid Argument

Suppose we want to prove some statement S .
A student assumes

$$\neg S$$

and also assumes

$$0 = 1.$$

They then say:

$$0 = 1$$

is impossible, so S is true.

This proves nothing.

The contradiction came from an independently false assumption, not from the negation of S .

TMUA Example 1

Which statement is equivalent to

$$P \rightarrow Q?$$

- (A) $Q \rightarrow P$
- (B) $\neg P \rightarrow \neg Q$
- (C) $\neg Q \rightarrow \neg P$
- (D) $P \wedge Q$

The answer is

(C).

TMUA Example 2

To prove

$$n^2 \text{ even} \Rightarrow n \text{ even}$$

by contrapositive, what should we assume?

- (A) n^2 odd
- (B) n odd
- (C) n^2 even
- (D) n even

We negate the conclusion.
Therefore:

TMUA Example 2 (continued)

(B).

TMUA Example 3

Suppose we want to prove:

$$x \neq 0.$$

What should a contradiction proof assume?

$$x = 0.$$

If this assumption leads, through valid reasoning, to an impossibility, then

$$x = 0$$

cannot hold.

Therefore

$$x \neq 0.$$

TMUA Example 4

A student wants to prove:

$$P \rightarrow Q.$$

They prove:

$$Q \rightarrow P.$$

Have they proved the original statement?

No.

They have proved the converse.

The logically equivalent indirect statement is

$$\neg Q \rightarrow \neg P.$$

TMUA Example 5

Suppose:

$$P \rightarrow Q.$$

Which assumption begins a proof by contrapositive?

$$\neg Q.$$

Which assumption begins a proof by contradiction of the implication?

Since

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q,$$

we assume

$$P \wedge \neg Q.$$

This distinction is important.

Negating an Implication

Recall:

$$P \rightarrow Q$$

fails only when

$$P$$

is true and

$$Q$$

is false.

Therefore:

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q.$$

So a contradiction proof of $P \rightarrow Q$ assumes

Negating an Implication (continued)

$$P \wedge \neg Q.$$

Compare the Two Methods

To prove

$$P \rightarrow Q :$$

Contrapositive:

Assume $\neg Q$ and prove $\neg P$.

Contradiction:

Assume $P \wedge \neg Q$ and derive an impossibility.

They are closely related, but the proof structures are not identical.

Problem 1

Prove:

$$n^2 \text{ divisible by } 3 \Rightarrow n \text{ divisible by } 3.$$

The contrapositive is:

$$3 \nmid n \Rightarrow 3 \nmid n^2.$$

If $3 \nmid n$, then

$$n \equiv 1 \text{ or } 2 \pmod{3}.$$

Therefore

$$n^2 \equiv 1^2 \text{ or } 2^2 \pmod{3},$$

so

$$n^2 \equiv 1 \pmod{3}.$$

Problem 1 (continued)

Hence $3 \nmid n^2$.

Problem 2

Suppose $n \in \mathbb{Z}$.

Prove:

$$n^2 + n \text{ is even.}$$

Factor:

$$n^2 + n = n(n + 1).$$

The integers n and $n + 1$ are consecutive.

One of two consecutive integers is even.

Therefore their product is even.

Hence:

$$2 \mid (n^2 + n).$$

What Method Did We Use?

The previous argument was direct.
We did not need contradiction.
We did not need contrapositive.
This matters.
The goal is not to use the most sophisticated method.
The goal is to use the clearest valid method.

Mathematical Maturity

Good mathematical reasoning includes choosing not only a correct method, but an efficient one.

Problem 3

Prove that there is no smallest positive real number.
Suppose, for contradiction, that r is the smallest positive real number.
Then

$$\frac{r}{2} > 0$$

and

$$\frac{r}{2} < r.$$

Thus $\frac{r}{2}$ is a positive real number smaller than r .
Contradiction.
Therefore:

There is no smallest positive real number.

Problem 4

Suppose x is irrational.
Prove that

$$-x$$

is irrational.
Assume, for contradiction, that

$$-x$$

is rational.
Then multiplying by -1 , which is rational, gives

$$x = -(-x),$$

which would be rational.
This contradicts the irrationality of x .
Hence:

Problem 4 (continued)

$-x$ is irrational.

Challenge 1

Write the contrapositive of:

$$x > 3 \Rightarrow x^2 > 9$$

for real x .

Challenge 2

Prove:

$$n^3 \text{ even} \Rightarrow n \text{ even.}$$

Challenge 3

Prove that there is no integer n satisfying

$$n < n < n + 1.$$

Challenge 4

Suppose $r \neq 0$ is rational and x is irrational.
Prove that

$$rx$$

is irrational.

Challenge 4 (continued)

Therefore:

rx is irrational.

Challenge 5

To prove

$$P \rightarrow (Q \wedge R)$$

by contrapositive, what should we assume?

Challenge 6

Negate:

$$\forall x \in \mathbb{R}, \quad f(x) > 0.$$

Challenge 7

Suppose $a, b \in \mathbb{Z}$.

Prove:

$$ab \text{ odd} \Rightarrow a \text{ odd and } b \text{ odd}.$$

Challenge 8

Prove that no real number satisfies

$$x^2 + 1 = 0.$$

Common Error 1: Proving the Converse

Original:

$$P \rightarrow Q.$$

Contrapositive:

$$\boxed{\neg Q \rightarrow \neg P}.$$

Converse:

$$Q \rightarrow P.$$

Do not confuse them.

The converse is not generally equivalent to the original.

Common Error 2: Incorrect Negation

The negation of

$$x > 5$$

is

$$x \leq 5,$$

not

$$x < 5.$$

The negation of

$$x = a$$

is

$$x \neq a.$$

Indirect proof depends on precise negation.

Common Error 3: Contradiction From an Error

If your algebra accidentally gives

$$1 = 0,$$

you have not necessarily proved anything.

Check whether every previous step was valid.

A contradiction proves the desired result only when it follows logically from the assumption being tested.

Common Error 4: Forgetting Original Hypotheses

In a contradiction proof, you retain the original assumptions.
You add the negation of the desired conclusion.
For an implication

$$P \rightarrow Q,$$

a contradiction proof therefore works under

$$\boxed{P \wedge \neg Q}.$$

Do not discard P .

Common Error 5: Declaring “Contradiction” Too Early

A surprising result is not necessarily a contradiction.
A contradiction must conflict with:

- an original assumption,
- an established fact,
- a definition,
- or another logically derived statement.

You should be able to state exactly:

$\boxed{\text{What contradicts what?}}$

Method 1: Direct Proof

Try direct proof when:

$$\boxed{P}$$

immediately gives useful algebraic or structural information.

Ask:

What follows naturally from the hypothesis?

If that chain leads efficiently to Q , use it.

Method 2: Contrapositive

Try contrapositive when:

$$\boxed{\neg Q}$$

is easier to represent than P .

This often happens with conclusions involving:

even/odd,

divisibility,

non-zero conditions,

or

absence of a property.

Method 3: Contradiction

Try contradiction when:

- the claim is a non-existence statement;
- the claim involves irrationality;
- assuming failure creates strong restrictions;
- a supposed extreme object can be improved;
- the opposite assumption naturally produces an impossibility.

Examples include:

$\sqrt{2}$ irrational,

no largest integer,

and

no smallest positive real.

A Proof-Method Decision Map

Can P naturally produce Q ?

If yes:

Direct Proof

Otherwise ask:

Is $\neg Q$ easier to use?

If yes:

Contrapositive

Otherwise ask:

Does assuming failure create an impossibility?

A Proof-Method Decision Map (continued)

Then consider:

Contradiction.

Proof Is About Structure

The same theorem may admit several proofs.
The important question is not merely:

Can I prove it?

but also:

Which proof exposes the structure most clearly?

A good proof often reveals why the result is inevitable.

The Negation Habit

Whenever you read a theorem, practise asking:

What is its exact negation?

For

$$P \rightarrow Q,$$

the negation is

$$P \wedge \neg Q.$$

For

$$\forall x P(x),$$

the negation is

$$\exists x \neg P(x).$$

The Negation Habit (continued)

For

$$\exists x P(x),$$

the negation is

$$\forall x \neg P(x).$$

What If the Statement Were False?

This is one of the most powerful questions in mathematics:

What would have to be true if this theorem were false?

That question transforms the problem.

Instead of trying to prove the theorem immediately, we study the consequences of its failure. Sometimes those consequences are impossible.

Mathematical Maturity

A developing student asks:

Which formula should I use?

A stronger problem solver asks:

What does the assumption force?

A mathematically mature problem solver also asks:

What would failure force?

That third question is the intellectual foundation of indirect proof.

Contrapositive

To prove

$$P \rightarrow Q,$$

we may instead prove

$$\neg Q \rightarrow \neg P.$$

These are logically equivalent:

$$(P \rightarrow Q) \iff (\neg Q \rightarrow \neg P).$$

Contradiction

To prove a statement S :

$$\text{Assume } \neg S.$$

Then derive an impossibility:

$$\perp.$$

Therefore:

$$\neg S$$

cannot hold.

Hence:

$$S.$$

Three Proof Routes

Method	Starting Point
Direct	P
Contrapositive	$\neg Q$
Contradiction	$\neg S$

For an implication $P \rightarrow Q$, contradiction assumes

$$\boxed{P \wedge \neg Q}.$$

Self-Check

Before moving on, make sure you can:

- 1 Write the contrapositive of an implication.
- 2 Distinguish contrapositive from converse.
- 3 Explain why contrapositive is equivalent to the original.
- 4 Construct a proof by contrapositive.
- 5 State the structure of proof by contradiction.
- 6 Negate inequalities correctly.
- 7 Negate conjunctions and disjunctions.
- 8 Negate quantified statements.
- 9 Explain the proof that $\sqrt{2}$ is irrational.
- 10 Choose sensibly between direct and indirect proof.

The TMUA Habit

When a direct proof looks awkward, ask:

What is the contrapositive?

Then ask:

Is its hypothesis easier to use?

If not, ask:

What happens if the result is false?

Then search for consequences that cannot coexist.

This is not a trick.

It is logical restructuring.

Coming in Part 8

Proof by Cases, Exhaustion and Counterexamples

Some problems cannot be handled efficiently by treating every possibility in the same way.
Instead, we divide the mathematical situation into

exhaustive cases

and prove the result separately in each case.

We will study:

Proof by Cases

Exhaustive Reasoning

Finite Possibility Arguments

Counterexamples and Disproof

and

How to Know Whether Your Cases Are Complete.

TMUA Paper 2 Masterclass

**Do Not Force the Direct Route.
Negate Precisely.
Follow What the Assumption Forces.**

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 8: Proof by Cases and Disproof by Counterexample Exhausting Possibilities and Breaking False Claims

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Two Very Different Mathematical Tasks

Sometimes we want to establish:

A statement is always true.

At other times we want to establish:

A statement is false.

These require very different reasoning.

To prove a universal statement, every possible case must be covered.

To disprove it, sometimes:

one carefully chosen example is enough.

The Asymmetry of Proof and Disproof

Suppose someone claims

$$\forall x \in D, \quad P(x).$$

Checking

$$P(x_1), P(x_2), P(x_3), \dots$$

for many examples does not normally prove the claim.

But finding just one $c \in D$ such that

$$\neg P(c)$$

immediately disproves it.

Thus:

Many confirming examples may not prove.

but

The Asymmetry of Proof and Disproof (continued)

One counterexample can disprove.

Part 8 Roadmap

We will study:

- ① Why proof by cases works
- ② Exhaustive cases
- ③ Even and odd cases
- ④ Positive, zero and negative cases
- ⑤ Absolute-value cases
- ⑥ Cases that overlap
- ⑦ Missing cases
- ⑧ What a counterexample is
- ⑨ Counterexamples to universal statements
- ⑩ Counterexamples to implications
- ⑪ Counterexamples to AND, OR, ONLY IF and IFF claims
- ⑫ Examples, conjectures and proof
- ⑬ TMUA-style reasoning

The Basic Idea

Suppose every possible object belongs to one of the cases

$$C_1, C_2, \dots, C_n.$$

If we prove the desired conclusion Q in every case,

$$C_1 \Rightarrow Q,$$

$$C_2 \Rightarrow Q,$$

$$\vdots$$

$$C_n \Rightarrow Q,$$

then Q holds in all possibilities under consideration.

The Cases Must Be Exhaustive

The crucial requirement is:

$$C_1 \vee C_2 \vee \dots \vee C_n$$

must cover every possible situation.

If even one legitimate possibility is omitted, the proof is incomplete.

Key Idea

Proof by cases works because the cases exhaust all possibilities, not merely because several examples have been checked.

The Simplest Example

Every integer is either

even

or

odd.

Therefore, to prove a statement for every integer n , it may be enough to establish:

Case 1: n is even

and

Case 2: n is odd.

These two cases are exhaustive.

Example 1

Prove that for every integer n ,

$$n(n+1) \text{ is even.}$$

We split into two cases.

Case 1: n is even

and

Case 2: n is odd.

Case 1: n Is Even

If n is even, then

$$n = 2k$$

for some integer k .

Therefore

$$n(n+1) = 2k(n+1).$$

Hence

$$n(n+1)$$

is divisible by 2.

Thus it is even.

Case 2: n Is Odd

If n is odd, then

$$n = 2k + 1.$$

Therefore

$$n + 1 = 2k + 2 = 2(k + 1).$$

Hence

$$n(n + 1) = n \cdot 2(k + 1),$$

which is divisible by 2.
Thus it is even.

Complete the Proof

Every integer is either even or odd.
We have shown:

$$n \text{ even} \Rightarrow n(n + 1) \text{ even}$$

and

$$n \text{ odd} \Rightarrow n(n + 1) \text{ even.}$$

Therefore:

$n(n + 1)$ is even for every integer n .

Could We Prove It More Directly?

Yes.

The integers

$$n \quad \text{and} \quad n + 1$$

are consecutive.

Among two consecutive integers, one is even.

Therefore their product is even.

This is shorter.

Mathematical Maturity

A proof by cases may be valid without being the most elegant proof. Always look for the underlying structure.

More Than Two Cases

Every integer has exactly one of the forms

$$3k, \quad 3k + 1, \quad 3k + 2.$$

Equivalently,

$$n \equiv 0, 1, \text{ or } 2 \pmod{3}.$$

These three possibilities can be used as exhaustive cases.

This is often useful in divisibility problems.

Example 2

Prove that for every integer n ,

$$n^2 \equiv 0 \text{ or } 1 \pmod{3}.$$

There are three possible cases:

$$n \equiv 0 \pmod{3},$$

$$n \equiv 1 \pmod{3},$$

or

$$n \equiv 2 \pmod{3}.$$

Check the Three Cases

If

$$n \equiv 0 \pmod{3},$$

then

$$n^2 \equiv 0.$$

If

$$n \equiv 1 \pmod{3},$$

then

$$n^2 \equiv 1.$$

If

$$n \equiv 2 \pmod{3},$$

Check the Three Cases (continued)

then

$$n^2 \equiv 4 \equiv 1 \pmod{3}.$$

Therefore:

$$n^2 \equiv 0 \text{ or } 1 \pmod{3}.$$

Why Residues Are Powerful

Instead of considering infinitely many integers

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots,$$

we classify them according to a finite structural property.

For modulo 3:

$$0, 1, 2.$$

An infinite problem has been reduced to three exhaustive cases.

Positive, Zero or Negative

Every real number x satisfies exactly one of:

$$x < 0,$$

$$x = 0,$$

$$x > 0.$$

These cases are exhaustive and mutually exclusive.

They are especially useful when expressions behave differently according to sign.

Example 3

Prove that

$$|x| \geq 0$$

for every real x .

By definition:

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Consider the two cases.

Case 1: $x \geq 0$

If

$$x \geq 0,$$

then

$$|x| = x.$$

Therefore

$$|x| \geq 0.$$

Case 2: $x < 0$

If

$$x < 0,$$

then

$$-x > 0.$$

Since

$$|x| = -x,$$

we obtain

$$|x| > 0.$$

Thus in either case,

$$\boxed{|x| \geq 0}.$$

Example 4

Show that

$$\boxed{|x|^2 = x^2}$$

for every real x .

If $x \geq 0$,

$$|x| = x,$$

so

$$|x|^2 = x^2.$$

If $x < 0$,

$$|x| = -x,$$

so

Example 4 (continued)

$$|x|^2 = (-x)^2 = x^2.$$

Therefore the identity holds for every real x .

Piecewise Definitions Suggest Cases

Whenever you see a definition such as

$$f(x) = \begin{cases} f_1(x), & x \in D_1, \\ f_2(x), & x \in D_2, \end{cases}$$

ask whether the natural proof should follow the same division:

$$x \in D_1$$

and

$$x \in D_2.$$

The definition itself may reveal the proof structure.

May Cases Overlap?

Yes.

For a proof by cases, the cases need to cover every possibility.

They do not always have to be mutually exclusive.

For example:

$$x \leq 0$$

and

$$x \geq 0$$

overlap at

$$x = 0,$$

but together they still cover every real number.

What Really Matters?

The essential requirement is:

Every allowed object must appear in at least one case.

Disjoint cases can make a proof cleaner.
But exhaustiveness is the logically essential property.

A Dangerous Case Split

Suppose a student wants to prove a result for every real x and considers:

Case 1: $x < 0$

and

Case 2: $x > 0$.

What is missing?

$x = 0$.

Even if both arguments are correct, the proof does not cover all real numbers.

Good Cases Follow Mathematical Structure

Do not split a problem randomly.

Useful case distinctions often arise from:

parity

sign

divisibility / residue

ordering

piecewise definitions

or

whether a key expression is zero.

Example: A Zero Case

Suppose an argument involves division by x .

A natural case split may be:

$$x = 0$$

and

$$x \neq 0.$$

Why?

Because division by x is valid only in the second case.

The case distinction protects the logical validity of the algebra.

Example 5

Solve over the reals:

$$x|x| = 4.$$

Consider:

$$x \geq 0$$

and

$$x < 0.$$

If $x \geq 0$,

$$|x| = x,$$

so

$$x^2 = 4.$$

Example 5 (continued)

Within this case,

$$x = 2.$$

Second Case

If

$$x < 0,$$

then

$$|x| = -x.$$

Therefore

$$x|x| = -x^2.$$

The equation becomes

$$-x^2 = 4,$$

or

$$x^2 = -4,$$

Second Case (continued)

which has no real solution.

Hence:

$$\boxed{x = 2}.$$

What Is a Counterexample?

A counterexample is an example that shows a statement is false.
Suppose the claim is:

$$\forall x \in D, \quad P(x).$$

A counterexample is some

$$c \in D$$

for which

$$\neg P(c).$$

One such example is sufficient to disprove the universal claim.

The Classic Structure

Claim:

$$\text{All objects in } D \text{ have property } P.$$

To disprove it, find one object c such that

$$c \in D$$

but

c does not have property P .

Then the universal statement is false.

Example 6

Claim:

All prime numbers are odd.

Consider

2.

We have:

2 is prime,

but

2 is even.

Therefore 2 is a counterexample.

Hence:

Example 6 (continued)

The claim is false.

Why One Example Is Enough

The claim

$$\forall x \in D, \quad P(x)$$

means that there are no exceptions.

Therefore discovering one exception immediately contradicts the claim.

Logically:

$$\neg(\forall x P(x))$$

is equivalent to

$$\boxed{\exists x \neg P(x)}.$$

Counterexample reasoning is therefore closely connected with quantifier negation.

A Crucial TMUA Pattern

Suppose the claim is

$$\boxed{A \Rightarrow B}.$$

When is an implication false?

Only when

$$\boxed{A \text{ is true}}$$

and

$$\boxed{B \text{ is false}}.$$

Therefore a counterexample to

$$A \Rightarrow B$$

must satisfy exactly this pattern.

Counterexample Template

To disprove

$$A(x) \Rightarrow B(x),$$

find x such that

$$A(x) \text{ is true}$$

but

$$B(x) \text{ is false}.$$

Do not merely find an x for which $B(x)$ is false.
The hypothesis must also hold.

Example 7

Disprove:

$$x < y \Rightarrow x^2 < y^2$$

for all real x, y .

Choose

$$x = -2, \quad y = 1.$$

Then

$$-2 < 1,$$

so the hypothesis is true.

But

$$(-2)^2 = 4 > 1 = 1^2.$$

Thus the conclusion is false.
Therefore the implication is false.

Why $x = 2, y = 1$ Is Not a Counterexample

For

$$x = 2, \quad y = 1,$$

the hypothesis

$$x < y$$

is false.

Therefore this pair does not demonstrate failure of

$$x < y \Rightarrow x^2 < y^2.$$

An implication with a false hypothesis is not a counterexample.

Common Mistake

To break $A \Rightarrow B$, you must make A true and B false.

Counterexample to $A \wedge B$

Suppose a claim says that for every relevant object,

$$A \wedge B.$$

To disprove it, we need:

$$\neg(A \wedge B).$$

By De Morgan's law,

$$\neg(A \wedge B) \equiv \neg A \vee \neg B.$$

So it is enough to find an example for which at least one of A and B fails.

Example

Claim:

Every prime is odd and greater than 4.

Take

3.

It is prime and odd,
but

$3 \not> 4$.

Thus the conjunction fails.
Therefore the universal claim is false.

Counterexample to $A \vee B$

Suppose the claim is:

$A \vee B$.

Its negation is

$\neg A \wedge \neg B$.

Therefore, to disprove an OR statement, we need an example where

A is false

and

B is false.

Both alternatives must fail.

Example

Claim:

Every positive integer is even or prime.

Consider

9.

It is not even.

It is also not prime.

Therefore both alternatives fail.

Hence:

9

is a counterexample.

Counterexample to “Only If”

Recall:

A only if B

means

$A \Rightarrow B$.

Therefore, to disprove it, find a situation where

$A = T$

and

$B = F$.

The same counterexample structure for implication applies.

Example

Claim:

$$n \text{ is odd only if } n \text{ is prime.}$$

This means:

$$n \text{ odd} \Rightarrow n \text{ prime.}$$

Take

$$n = 9.$$

Then 9 is odd,
but 9 is not prime.
Therefore:

$$9$$

is a counterexample.

Counterexample to an IFF Claim

The statement

$$A \iff B$$

requires both:

$$A \Rightarrow B$$

and

$$B \Rightarrow A.$$

Therefore it can be disproved by breaking either direction.
We need an example where A and B have different truth values.

Example

Claim:

n is prime if and only if n is odd.

Take

$$n = 9.$$

Then:

n is odd

but

n is not prime.

Thus one direction fails.
Alternatively,

Example (continued)

$$n = 2$$

is prime but not odd.
Either example disproves the equivalence.

Checking Examples

Suppose we observe:

$$1^2 + 1 = 2,$$

$$2^2 + 2 = 6,$$

$$3^2 + 3 = 12,$$

$$4^2 + 4 = 20.$$

Every result is even.

This suggests the conjecture:

$$n^2 + n \text{ is even for every integer } n.$$

But the examples themselves do not prove it.

From Pattern to Conjecture

Small cases are valuable because they can help us:

notice a pattern



form a conjecture



search for a proof.

For the previous example:

$$n^2 + n = n(n + 1).$$

From Pattern to Conjecture (continued)

Two consecutive integers always include an even integer.
Therefore the conjecture is proved.

The Correct Role of Examples

Examples can:

- suggest a conjecture;
- reveal structure;
- expose a false conjecture;
- help us design a proof;
- test whether an argument is plausible.

But examples alone normally do not establish a universal mathematical theorem.

One-Sided Power of Examples

There is an important asymmetry:

Examples are weak evidence for universal truth

but

a counterexample is decisive evidence of universal falsehood.

This is why actively searching for counterexamples is such an important mathematical habit.

Do Not Search Randomly

Suppose we want to disprove:

$$x < y \Rightarrow x^2 < y^2.$$

Why might it fail?

Squaring depends not only on order but also on magnitude and sign.

This suggests testing:

negative values

or values on opposite sides of zero.

That leads quickly to:

$$x = -2, \quad y = 1.$$

A Counterexample Search Strategy

When testing a universal claim, try:

boundary values

such as 0, 1, or equality cases;

negative values

when signs matter;

small integers

when divisibility or primality matters;

extreme values

when inequalities are involved;
and

A Counterexample Search Strategy (continued)

special structural cases.

Example 8

Claim:

$$a^2 > b^2 \Rightarrow a > b$$

for all real a, b .

We want:

$$a^2 > b^2$$

but

$$a \leq b.$$

Try a large negative a and a small positive b :

$$a = -3, \quad b = 2.$$

Then

Example 8 (continued)

$$9 > 4$$

but

$$-3 \not> 2.$$

Thus the claim is false.

When Checking Every Possibility Is a Proof

There is one important exception to the statement

“checking examples is not proof.”

If the domain contains only finitely many possibilities and you genuinely check all of them, then the argument can be a proof by exhaustive cases.

Example 9

Suppose n is an integer satisfying

$$1 \leq n \leq 4.$$

Prove:

$$n^2 + n$$

is even.

There are only four possibilities:

$$n = 1, 2, 3, 4.$$

Checking all four would establish the result for this restricted domain.
That is exhaustive reasoning, not mere experimentation.

Why the Domain Matters

Checking

$$n = 1, 2, 3, 4$$

does not prove the statement for all integers.
But if the hypothesis explicitly restricts n to

$$\{1, 2, 3, 4\},$$

then those are all possible cases.

Key Idea

The difference between “examples” and “exhaustion” depends on whether every allowed possibility has actually been covered.

TMUA Example 1

A student wants to prove a statement for every integer.
They establish it for:

$$n \text{ even}$$

and

$$n \text{ odd.}$$

Assuming both arguments are correct, is the proof complete?

Yes.

Every integer is either even or odd.
The cases are exhaustive.

TMUA Example 2

A student wants to prove a statement for every real number.
They establish it for:

$$x < 0$$

and

$$x > 0.$$

Is the proof complete?

No.

The case

$$x = 0$$

has not been considered.

TMUA Example 3

To disprove

$$A \Rightarrow B,$$

which truth-value combination is required?

(A) $A = F, B = F$

(B) $A = F, B = T$

(C) $A = T, B = F$

(D) $A = T, B = T$

The answer is

(C).

TMUA Example 4

Claim:

Every integer greater than 1 is prime or even.

Which number is a counterexample?

- (A) 2 (B) 3 (C) 6 (D) 9

For 9:

9 is not prime

and

9 is not even.

Therefore:

(D)

TMUA Example 4 (continued)

is a counterexample.

TMUA Example 5

Suppose a claim says:

$$\forall x \in \mathbb{R}, \quad f(x) > 0.$$

What constitutes a counterexample?

We need some real c such that

$$f(c) > 0$$

is false.

Therefore:

$$f(c) \leq 0.$$

Notice the boundary case $f(c) = 0$ is sufficient.

TMUA Example 6

A student checks a formula for

$$n = 1, 2, \dots, 1000$$

and it works every time.

Have they proved that it holds for every positive integer?

No.

Unless there is an argument showing why all remaining positive integers must behave the same way, infinitely many cases remain.

Problem 1

Prove that for every integer n ,

$$n^2 + n + 2 \text{ is even.}$$

Case 1: n is even.

Let

$$n = 2k.$$

Then

$$n^2 + n + 2 = 4k^2 + 2k + 2 = 2(2k^2 + k + 1).$$

Hence it is even.

Problem 1: Odd Case

Case 2: n is odd.

Let

$$n = 2k + 1.$$

Then

$$n^2 + n + 2 = (2k + 1)^2 + (2k + 1) + 2$$

$$= 4k^2 + 6k + 4$$

$$= 2(2k^2 + 3k + 2).$$

Hence it is even.

Since every integer is even or odd, the result follows.

Problem 2

Prove that for every real x ,

$$\boxed{|x| \geq x}.$$

Case 1:

$$x \geq 0.$$

Then

$$|x| = x,$$

so equality holds.

Case 2:

$$x < 0.$$

Then

Problem 2 (continued)

$$|x| = -x.$$

Since $x < 0$,

$$-x > x.$$

Thus the result holds in both cases.

Problem 3

Disprove:

$$a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$$

for all non-zero real a, b .

Choose values on opposite sides of zero:

$$a = -1, \quad b = 1.$$

Then

$$-1 < 1,$$

but

$$\frac{1}{-1} = -1$$

and

Problem 3 (continued)

$$\frac{1}{1} = 1.$$

So

$$-1 \not> 1.$$

The implication is false.

What Went Wrong With the Claim?

The familiar rule

$$a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$$

requires additional sign conditions.

For example, it works when

$$0 < a < b.$$

The counterexample does more than disprove the original claim.

It reveals a missing hypothesis.

Mathematical Maturity

A good counterexample can diagnose why a theorem fails and suggest how it should be repaired.

Failure Can Improve a Statement

Suppose we conjecture:

$$x < y \Rightarrow x^2 < y^2.$$

The counterexample

$$x = -2, \quad y = 1$$

shows the claim is false.

But now ask:

What condition would make it true?

For example, if

$$0 \leq x < y,$$

then

Failure Can Improve a Statement (continued)

$$x^2 < y^2.$$

The Counterexample Cycle

A powerful mathematical workflow is:

Conjecture



Search for counterexamples



If one exists:

Diagnose the failure



The Counterexample Cycle (continued)

Refine the conjecture



Prove the improved statement.

Challenge 1

Prove that for every integer n ,

$$n^2 + n$$

is even using even and odd cases.

Challenge 2

Are the cases

$$x \leq 2$$

and

$$x \geq 2$$

exhaustive over $\mathbb{R}$?

Challenge 3

Are the cases

$$x < 2$$

and

$$x > 2$$

exhaustive over $\mathbb{R}$?

Challenge 4

Find a counterexample to:

$$x^2 > x$$

for every positive real x .

Challenge 5

Find a counterexample to:

$$n \text{ odd} \Rightarrow n \text{ prime.}$$

Challenge 6

What must be true of a counterexample to

$$A \vee B?$$

Challenge 7

What must be true of a counterexample to

$$A \iff B?$$

Challenge 8

A student checks that

$$n^2 - n + 41$$

is prime for many small positive integers n .
What may they conclude?

Common Error 1: Missing a Case

A proof covers

$$x < 0$$

and

$$x > 0$$

but forgets

$$x = 0.$$

Even if both completed cases are correct,

the proof is incomplete.

Always verify exhaustiveness before doing the algebra.

Common Error 2: Checking Several Examples

A student checks

$$n = 1, 2, 3, 4, 5$$

and concludes a result holds for every integer.

This is not a proof.

The unchecked domain is still infinite.

Examples can suggest.

Proof must justify.

Common Error 3: Bad Counterexample

To disprove

$$A \Rightarrow B,$$

a student chooses an example for which

$$A = F, \quad B = F.$$

This does not disprove the implication.

The required pattern is:

$A = T, \quad B = F.$

Common Error 4: Leaving the Domain

Claim:

$$\forall x > 0, \quad P(x).$$

A student finds $x = -2$ for which $P(x)$ is false.

This is not a counterexample.

Why?

Because

$$-2$$

does not satisfy the hypothesis

$$x > 0.$$

A counterexample must belong to the stated domain.

Common Error 5: One Example “Proves” a Claim

Suppose:

$$P(2)$$

is true.

This proves only

$$P(2).$$

It does not prove

$$\forall x P(x).$$

Do not confuse:

verification

with

Common Error 5: One Example “Proves” a Claim (continued)

proof.

Common Error 6: Random Case Splitting

A case split should simplify the mathematics.

Dividing into

$$x < 100$$

and

$$x \geq 100$$

is exhaustive,

but may be useless if nothing important changes at 100.

Good cases reflect the structure of the problem.

When You See “For All”

When a claim says

for all ,

immediately ask two different questions:

Can I prove it structurally?

and

Can I break it with one example?

Before investing in a long proof, test the statement intelligently.

Before Proving, Stress-Test

Suppose you encounter a new conjecture.

Try:

0, 1, -1 ,

small positive and negative values,

boundary cases,

equality cases,

and structurally unusual examples.

If the statement survives, then search for a proof.

If it fails, understand why.

When You See a Natural Split

Ask:

Does the mathematics change depending on a condition?

Examples:

n even or odd,

$x \geq 0$ or $x < 0$,

$x = 0$ or $x \neq 0$,

$n \equiv 0, 1, 2 \pmod{3}$.

If so, proof by cases may be natural.

The Two Questions

For proof by cases:

Have I covered every possibility?

For counterexamples:

Does my example satisfy the hypothesis and violate the conclusion?

These two checks prevent most elementary errors in this topic.

Proof and Disproof Are Not Symmetric

To establish

$$\forall x P(x),$$

you need reasoning that covers every x .

To establish

$$\neg \forall x P(x),$$

you need only find

$$\exists x \neg P(x).$$

Thus the logical structure of a statement determines what kind of evidence is sufficient.

Learn to Attack Your Own Conjectures

When you believe a statement is true, do not search only for supporting examples.
Actively try to destroy it.

Ask:

What happens at the boundary?

What happens for negative values?

What happens when something becomes zero?

What is the smallest unusual case?

A Counterexample Is Information

A counterexample does not merely say:

Wrong.

A useful counterexample can reveal:

which assumption is missing

where the argument breaks

how the theorem should be modified

and sometimes

what the correct theorem should be .

From Experiment to Mathematics

A mature workflow is:

Explore examples



Recognise structure



Form a conjecture



From Experiment to Mathematics (continued)

Try to break it



Refine if necessary



Prove the final statement.

Proof by Cases

A proof by cases has the structure:

$$C_1 \vee C_2 \vee \cdots \vee C_n$$

together with

$$C_1 \Rightarrow Q,$$

$$C_2 \Rightarrow Q,$$

$\vdots$

$$C_n \Rightarrow Q.$$

Therefore:

Proof by Cases (continued)

$\square Q$.

The cases must collectively cover all possibilities.

Disproof by Counterexample

To disprove

$$\forall x \in D, \quad P(x),$$

find

$$c \in D$$

such that

$$\neg P(c).$$

Then:

$$\forall x \in D, \quad P(x)$$

is false.

Counterexample to an Implication

To disprove

$$A \Rightarrow B,$$

find an example satisfying:

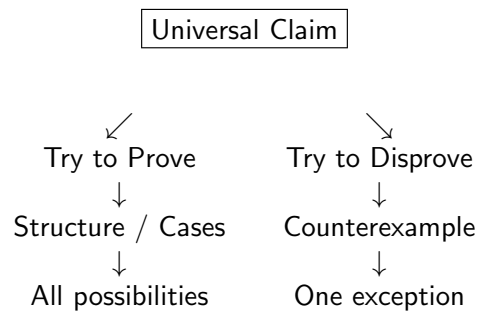
$$A = T$$

and

$$B = F.$$

No other truth-value combination makes an implication false.

Concept Map



Understanding both directions is essential mathematical reasoning.

Self-Check

Before moving on, make sure you can:

- ① Explain why proof by cases is valid.
- ② Determine whether a set of cases is exhaustive.
- ③ Use even/odd cases.
- ④ Use sign cases.
- ⑤ Use residue cases.
- ⑥ Handle piecewise definitions using cases.
- ⑦ Define a counterexample.
- ⑧ Disprove a universal statement.
- ⑨ Construct a counterexample to an implication.
- ⑩ Explain why examples do not normally prove universal claims.
- ⑪ Distinguish finite exhaustion from experimental checking.
- ⑫ Use counterexamples to refine conjectures.

The TMUA Habit

When you think a statement is true:

Try to prove it.

But also:

Try to break it.

When you divide into cases:

Check that nothing is missing.

When you claim a counterexample:

Check that it really makes the statement false.

This combination of constructive and critical reasoning is central to mathematical maturity.

Coming in Part 9

Conjectures, Implications and Chains of Reasoning

TMUA Paper 2 does not stop at recognising standard proof methods.
You must also be able to move through a sequence of mathematical statements and determine

what actually follows from what.

We will develop:

Making Conjectures from Small Cases

Justifying Conjectures

Coming in Part 9 (continued)

Deducing Implications

Ordering Steps of a Proof

and

Sophisticated Chains of Reasoning.

**Cover Every Case to Prove.
Find One Exception to Disprove.
Know Which Task You Are Doing.**

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 9: Conjectures, Implications and Chains of Reasoning
From Individual Facts to Complete Mathematical Arguments

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Beyond Individual Proof Techniques

So far we have studied tools such as:

direct proof

contrapositive

contradiction

proof by cases

and

counterexample.

But mathematical reasoning is not merely about knowing the names of proof methods. The harder skill is deciding:

Beyond Individual Proof Techniques (continued)

What follows from the information I have?

The Central Idea of Part 9

A mathematical argument often looks like

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E.$$

The challenge is not usually performing one enormous step. Instead, it is recognising a sequence of smaller valid steps.

Key Idea

Complex reasoning is often a chain of simple implications connected in the correct order.

What We Will Learn

In this part we will develop four major skills:

- ① Deduce implications from given information.
- ② Make conjectures from small cases and then justify them.
- ③ Rearrange mathematical statements into a valid proof.
- ④ Construct and analyse longer chains of reasoning.

These skills bring together almost everything we have studied in Paper 2 so far.

A Note on Notation

Throughout this lesson we will often write

$$A \Rightarrow B$$

as a compact teaching notation for

if A , then B .

Similarly, chains such as

$$A \Rightarrow B \Rightarrow C$$

will help us see the structure of an argument.

TMUA Thinking

The important skill is understanding the logical relationship between the statements, not merely manipulating symbols.

What Does It Mean to Deduce?

Suppose we know:

$$A \Rightarrow B$$

and

$$A$$

is true.

Then we may conclude:

$$B.$$

This is the simplest form of deduction.

We begin with information known to be true and obtain a new statement through a valid implication.

Example 1

Suppose:

n is divisible by 12.

What can we deduce?

Since

$$12 = 3 \cdot 4,$$

we can deduce:

$$3 \mid n$$

and

$$4 \mid n.$$

Since $4 \mid n$, we may further deduce:

Example 1 (continued)

$$2 \mid n.$$

Thus:

$$12 \mid n \Rightarrow 4 \mid n \Rightarrow 2 \mid n.$$

A Chain of Consequences

The information

$$12 \mid n$$

does not give us only one consequence.
It generates a network of consequences:

$$12 \mid n \Rightarrow 6 \mid n,$$

$$12 \mid n \Rightarrow 4 \mid n,$$

$$12 \mid n \Rightarrow 3 \mid n,$$

$$12 \mid n \Rightarrow 2 \mid n.$$

A strong problem solver asks:

A Chain of Consequences (continued)

What useful consequences can I extract?

Do Not Reverse the Information

From

$$12 \mid n \Rightarrow 3 \mid n$$

we cannot conclude

$$3 \mid n \Rightarrow 12 \mid n.$$

For example:

$$n = 6$$

is divisible by 3,
but not by 12.

Common Mistake

A valid implication may generate useful consequences, but its converse does not automatically become available.

The Fundamental Chain

Suppose:

$$A \Rightarrow B$$

and

$$B \Rightarrow C.$$

Then:

$$\boxed{A \Rightarrow C}.$$

Why?

If A is true, then B must be true.

Once B is true, C must be true.

Therefore:

$$A \Rightarrow B \Rightarrow C.$$

Extending the Chain

If

$$A \Rightarrow B,$$

$$B \Rightarrow C,$$

$$C \Rightarrow D,$$

then

$$A \Rightarrow B \Rightarrow C \Rightarrow D.$$

Therefore:

$$A \Rightarrow D.$$

Each intermediate statement acts as a bridge to the next one.

Example 2

Suppose:

n is divisible by 8.

Then

$$n = 8k$$

for some integer k .

Therefore:

$$n = 2(4k),$$

so n is even.

Hence:

$$n^2 \text{ is even.}$$

Thus we have the chain

Example 2 (continued)

$$8 \mid n \Rightarrow n \text{ even} \Rightarrow n^2 \text{ even.}$$

Intermediate Statements Matter

Suppose our target is:

$$A \Rightarrow D.$$

Sometimes the connection between A and D is not obvious.

The real problem is to discover:

$$B \quad \text{and} \quad C$$

such that

$$A \Rightarrow B \Rightarrow C \Rightarrow D.$$

Mathematical Maturity

A sophisticated proof often depends on discovering the right intermediate statements.

Forward Reasoning

Suppose we are given:

$$A$$

and asked to establish Z .
One strategy is to ask:

What follows from A ?

Perhaps:

$$A \Rightarrow B.$$

Then ask:

What follows from B ?

Perhaps:

Forward Reasoning (continued)

$$B \Rightarrow C.$$

Continue until the target appears.

Forward Chain

The process becomes:

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow \cdots \Rightarrow Z.$$

This is especially effective when the hypothesis has a strong algebraic or structural consequence. Examples include:

divisibility,

parity,

factorisation,

inequalities,

and

Forward Chain (continued)

geometric conditions.

Example 3

Suppose n is an odd integer.
Show that

$$n^2$$

is odd.

Start from the hypothesis:

$$n \text{ odd.}$$

Therefore:

$$n = 2k + 1$$

for some integer k .

Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$$

Example 3 (continued)

Thus:

$$n^2 = 2(2k^2 + 2k) + 1.$$

Therefore:

$$\boxed{n^2 \text{ is odd.}}$$

Reasoning from the Target

Sometimes the hypothesis does not immediately suggest what to do.

Then look at the conclusion.

Ask:

What would be sufficient to establish the target?

If the target is Z , perhaps it would be enough to prove Y :

$$Y \Rightarrow Z.$$

Then ask what would establish Y .

Backward Planning

We may think:

$$A \quad ? \quad B \quad ? \quad C \quad ? \quad Z.$$

Working backward from Z , we identify what would be sufficient:

$$C \Rightarrow Z.$$

Then:

$$B \Rightarrow C.$$

Then:

$$A \Rightarrow B.$$

The final written proof still proceeds forward:

$$A \Rightarrow B \Rightarrow C \Rightarrow Z.$$

Discovery vs Presentation

This distinction is important.

You may **discover** a proof by reasoning backward.

But you may **present** it forward.

For example:

$$\text{Target} \leftarrow \text{needed condition} \leftarrow \text{earlier condition}$$

during exploration,
then write:

$$\text{Hypothesis} \Rightarrow \text{intermediate steps} \Rightarrow \text{Target}.$$

What Is a Conjecture?

A conjecture is a mathematical statement that we suspect is true, usually because we have observed a pattern or discovered supporting evidence.

For example, calculate:

$$1 + 3 = 4,$$

$$1 + 3 + 5 = 9,$$

$$1 + 3 + 5 + 7 = 16.$$

A pattern appears:

$$4 = 2^2, \quad 9 = 3^2, \quad 16 = 4^2.$$

What might we conjecture?

Forming the Conjecture

The pattern suggests:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

So our conjecture is:

$$\sum_{k=1}^n (2k - 1) = n^2.$$

But observing the pattern is only the first stage.
We still need justification.

Small Cases: What They Can Do

Small cases can help us:

- recognise a pattern;
- guess a general formula;
- discover a possible invariant;
- identify exceptional cases;
- test whether a conjecture is plausible;
- suggest how a proof might work.

But unless the domain has been exhausted:

small cases are not themselves a proof.

Example 4: Discovering a Pattern

Calculate:

$$1^3 = 1,$$

$$1^3 + 2^3 = 9,$$

$$1^3 + 2^3 + 3^3 = 36,$$

$$1^3 + 2^3 + 3^3 + 4^3 = 100.$$

Now notice:

$$1 = 1^2,$$

$$9 = 3^2,$$

Example 4: Discovering a Pattern (continued)

$$36 = 6^2,$$

$$100 = 10^2.$$

What are

$$1, 3, 6, 10?$$

A Possible Conjecture

We recognise:

$$1 = 1,$$

$$3 = 1 + 2,$$

$$6 = 1 + 2 + 3,$$

$$10 = 1 + 2 + 3 + 4.$$

This suggests:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Equivalently:

A Possible Conjecture (continued)

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

The pattern generates the conjecture.
It does not yet prove it.

Do Not Become Attached to a Pattern

Suppose the first several cases support a statement.
A strong mathematician does not immediately decide:

It must be true.

Instead ask:

Can I find a counterexample?

The purpose of small cases is exploration, not confirmation bias.

A Famous Warning

Consider:

$$p(n) = n^2 + n + 41.$$

For many small non-negative integers n , this expression gives prime numbers.
For example:

$$p(0) = 41,$$

$$p(1) = 43,$$

$$p(2) = 47.$$

It is tempting to conjecture that $p(n)$ is always prime.
But consider $n = 40$.

The Pattern Eventually Breaks

At

$$n = 40,$$

we obtain

$$p(40) = 40^2 + 40 + 41.$$

Therefore

$$p(40) = 1681.$$

But

$$1681 = 41^2.$$

Hence it is not prime.

So even a long sequence of successful small cases need not prove a universal claim.

The Correct Conjecture Workflow

A useful reasoning process is:

Generate small cases



Recognise a pattern



Form a conjecture



The Correct Conjecture Workflow (continued)

Stress-test it



Justify or revise it.

Pattern Recognition Is Only Stage One

Suppose we observe:

$$1(2) = 2,$$

$$2(3) = 6,$$

$$3(4) = 12,$$

$$4(5) = 20.$$

All are even.

We conjecture:

$n(n + 1)$ is even for every integer n .

How can we justify it?

Structural Justification

The numbers

$$n \quad \text{and} \quad n + 1$$

are consecutive integers.

Of any two consecutive integers, exactly one is even.

Therefore one of the factors in

$$n(n + 1)$$

is even.

Hence:

$$n(n + 1) \text{ is even.}$$

The examples suggested the theorem.

Structure proved it.

The Important Transition

The intellectual transition is:

"It seems to work"

to

"Here is why it must work."

That transition is one of the central ideas of mathematical proof.

A Proof as a Chain

Suppose we want to prove:

$$A \Rightarrow Z.$$

A proof might have the structure:

$$A \Rightarrow P,$$

$$P \Rightarrow Q,$$

$$Q \Rightarrow R,$$

$$R \Rightarrow Z.$$

Thus:

$$A \Rightarrow P \Rightarrow Q \Rightarrow R \Rightarrow Z.$$

Every Link Must Be Valid

A proof chain is only as strong as its weakest link.

Suppose:

$$A \Rightarrow B$$

is valid,

$$B \Rightarrow C$$

is valid,
but

$$C \Rightarrow D$$

is invalid.

Then the complete chain

$$A \Rightarrow B \Rightarrow C \Rightarrow D$$

Every Link Must Be Valid (continued)

does not establish D .

Key Idea

Every transition in a mathematical proof requires justification.

Example 5

Prove:

$$12 \mid n \Rightarrow n^2 \text{ is divisible by } 16.$$

Start with:

$$12 \mid n.$$

Then

$$n = 12k$$

for some integer k .

Hence:

$$n^2 = 144k^2.$$

Since

Example 5 (continued)

$$144 = 16 \cdot 9,$$

we have

$$n^2 = 16(9k^2).$$

Therefore:

$$\boxed{16 \mid n^2}.$$

Identify the Logical Skeleton

The previous algebra can be viewed structurally:

$$12 \mid n$$

$$\Downarrow$$

$$n = 12k$$

$$\Downarrow$$

$$n^2 = 144k^2$$

$$\Downarrow$$

Identify the Logical Skeleton (continued)

$$n^2 = 16(9k^2)$$

$\Downarrow$

$$16 \mid n^2.$$

The algebra and the logic are working together.

Scrambled Proofs

Suppose a proof contains the correct statements but they have been placed in the wrong order.
Your task is to reconstruct the dependency chain.

The key question is:

Which statement provides the information needed for the next?

This is not merely a language exercise.
It tests whether you understand the logical architecture of a proof.

Example 6: Scrambled Statements

We want to prove:

$$n \text{ even} \Rightarrow n^2 \text{ divisible by } 4.$$

The following statements are scrambled:

$$(P) \quad n^2 = 4k^2,$$

$$(Q) \quad n = 2k \text{ for some integer } k,$$

$$(R) \quad 4 \mid n^2,$$

$$(S) \quad n \text{ is even.}$$

What is the correct order?

Reconstructing the Proof

Start from the hypothesis:

S

because n is even.

Then by the definition of even:

Q

so $n = 2k$.

Squaring gives:

P

because

$$n^2 = 4k^2.$$

Finally:

Reconstructing the Proof (continued)

R

because n^2 is divisible by 4.
Thus:

$$S \rightarrow Q \rightarrow P \rightarrow R.$$

How to Order a Proof

Look for three special roles.
First:

Starting statement

usually comes directly from the hypothesis.
Second:

Intermediate statements

are consequences of earlier information.
Third:

Final statement

should match the required conclusion.
Then determine the dependency between the intermediate steps.

A Gap in an Argument

Consider:

n is divisible by 6

$\Downarrow$

n^2 is divisible by 36.

The conclusion is true.

But suppose we want to make the reasoning explicit.

What intermediate step could we insert?

$$6 \mid n \Rightarrow \boxed{n = 6k} \Rightarrow n^2 = 36k^2 \Rightarrow 36 \mid n^2.$$

Why Missing-Step Questions Matter

A missing step must satisfy two conditions.

It must follow from the previous statement:

$$P \Rightarrow \boxed{?}$$

and it must be strong enough to produce the next statement:

$$\boxed{?} \Rightarrow Q.$$

So we are looking for a logical bridge:

$$\boxed{P \Rightarrow B \Rightarrow Q.}$$

Example 7

Complete the chain:

$$x > 3 \Rightarrow \boxed{?} \Rightarrow x^2 > 9.$$

A possible intermediate statement is:

$$x > 0.$$

But is that enough?

No.

From

$$x > 0$$

alone we cannot conclude

$$x^2 > 9.$$

For example $x = 1$.

So the missing statement must be sufficiently strong.

A Better Bridge

From

$$x > 3$$

we already know:

$$x^2 > 3^2$$

because both sides are positive and squaring preserves the ordering here.

Thus:

$$x > 3 \Rightarrow x^2 > 3^2 \Rightarrow x^2 > 9.$$

The intermediate statement must not merely be true.

It must actually advance the proof.

Not Every Given Fact Must Be Useful

A problem may give several statements:

$A, B, C, D.$

The target may follow only from

A and $C.$

Do not assume that every piece of information must appear in your argument.

The real question is:

Which information is logically relevant to the target?

Example 8

Suppose we know:

n is divisible by 12,

$n > 100,$

and

n is a square.

We are asked to prove:

n is even.

Which information is needed?

Only:

$12 \mid n.$

Example 8 (continued)

Because:

$$12 \mid n \Rightarrow 2 \mid n.$$

The other facts are irrelevant to this particular conclusion.

Deduction Is Also About Restraint

Strong reasoning is not only knowing what follows.

It is also knowing what does *not* follow.

Suppose:

$$A \Rightarrow B$$

and

$$C \Rightarrow B.$$

If B is true, can we conclude A ?

No.

Can we conclude C ?

No.

There may be many different ways for B to be true.

Example 9

Suppose:

$$4 \mid n \Rightarrow n \text{ even}$$

and

$$6 \mid n \Rightarrow n \text{ even.}$$

We are told:

$$n \text{ is even.}$$

Can we conclude:

$$4 \mid n?$$

No.

Can we conclude:

Example 9 (continued)

$$6 \mid n?$$

No.

For example:

$$n = 2.$$

The Direction of Information

Implications have direction.
If we know

$$A \Rightarrow B,$$

the arrow tells us how information flows:

$$\boxed{A \longrightarrow B}.$$

It does not automatically flow backward:

$$B \nRightarrow A.$$

When constructing a chain, always respect the direction of each logical link.

Not Every Argument Is a Straight Line

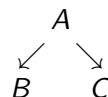
Sometimes information branches.
Suppose:

$$A \Rightarrow B$$

and

$$A \Rightarrow C.$$

Then from A we obtain both B and C :



Perhaps the final conclusion requires both.

Combining Branches

Suppose:

$$A \Rightarrow B,$$

$$A \Rightarrow C,$$

and

$$B \wedge C \Rightarrow D.$$

Then:

$$A \Rightarrow (B \wedge C) \Rightarrow D.$$

So a sophisticated argument may involve both:

splitting

and later

Combining Branches (continued)

recombining information.

Example 10

Suppose:

$$12 \mid n.$$

Then:

$$3 \mid n$$

and

$$4 \mid n.$$

So from one hypothesis we obtain two consequences:

$$12 \mid n \Rightarrow \begin{cases} 3 \mid n, \\ 4 \mid n. \end{cases}$$

Different later parts of a problem may use different branches.

Conditions Can Accumulate

Suppose:

$$A \Rightarrow B$$

and

$$A \Rightarrow C.$$

Then under A , both B and C hold.

So we may conclude:

$$\boxed{A \Rightarrow (B \wedge C).}$$

This can be useful when a later result requires two conditions simultaneously.

Example 11

Suppose n is divisible by 12.

Then:

$$3 \mid n$$

and

$$4 \mid n.$$

Therefore:

$$12 \mid n \Rightarrow (3 \mid n \text{ and } 4 \mid n).$$

But be careful with the reverse direction in other settings.

Combining conditions requires checking the relevant arithmetic structure.

Example 12

Suppose:

$$x + \frac{1}{x} = 3, \quad x \neq 0.$$

Find:

$$x^2 + \frac{1}{x^2}.$$

The target is related to the square of the given expression.

So square both sides:

$$\left(x + \frac{1}{x}\right)^2 = 9.$$

Continue the Chain

Expanding:

$$x^2 + 2 + \frac{1}{x^2} = 9.$$

Therefore:

$$x^2 + \frac{1}{x^2} = 7.$$

The reasoning chain is:

$$x + \frac{1}{x} = 3$$

$\Downarrow$

$$\left(x + \frac{1}{x}\right)^2 = 9$$

Continue the Chain (continued)

$\Downarrow$

$$x^2 + 2 + \frac{1}{x^2} = 9$$

$\Downarrow$

$$\boxed{x^2 + \frac{1}{x^2} = 7}.$$

Why This Is a Reasoning Problem

The algebra itself is elementary.

The important decision was recognising that the target

$$x^2 + \frac{1}{x^2}$$

appears naturally when we square

$$x + \frac{1}{x}.$$

Thus the key step was not calculation.

It was:

choosing the useful transformation.

Example 13

Suppose

$$x + y = 10$$

and

$$xy = 21.$$

Find:

$$x^2 + y^2.$$

We know:

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Therefore:

$$100 = x^2 + y^2 + 42.$$

Example 13 (continued)

Hence:

$$x^2 + y^2 = 58.$$

Now Go One Step Further

Using

$$x^2 + y^2 = 58$$

and

$$xy = 21,$$

find:

$$(x - y)^2.$$

Since

$$(x - y)^2 = x^2 + y^2 - 2xy,$$

we obtain:

$$(x - y)^2 = 58 - 42 = 16.$$

Now Go One Step Further (continued)

Thus:

$$(x - y)^2 = 16.$$

The Dependency Structure

Notice how the second conclusion depends on the first:

$$x + y = 10, \quad xy = 21$$

$$\Downarrow$$

$$x^2 + y^2 = 58$$

$$\Downarrow$$

$$(x - y)^2 = 16.$$

One derived result becomes input for the next stage.
This is typical of longer reasoning problems.

Example 14

Suppose we know:

$$a + b = 7$$

and

$$a^2 + b^2 = 29.$$

Find ab .

We need a relationship containing all three quantities.

Use:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Substitute:

$$49 = 29 + 2ab.$$

Therefore:

Example 14 (continued)

$$\boxed{ab = 10}.$$

The Hidden Bridge

The given information does not directly contain ab .
The target is reached through the identity:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

This identity is the bridge connecting:

what we know

to

what we want.

Recognising such bridges is central to sophisticated reasoning.

Example 15

Suppose we want to prove:

$$n \text{ odd} \Rightarrow n^2 \equiv 1 \pmod{8}.$$

The following steps are scrambled:

- (A) $k(k + 1)$ is even,
- (B) $n = 2k + 1$,
- (C) $n^2 = 4k(k + 1) + 1$,
- (D) $8 \mid 4k(k + 1)$,
- (E) $n^2 \equiv 1 \pmod{8}$.

What is the logical order?

Reconstruct the Chain

From n odd:

B

because

$$n = 2k + 1.$$

Squaring and rearranging gives:

C

since

$$n^2 = 4k(k + 1) + 1.$$

Because k and $k + 1$ are consecutive:

A .

Reconstruct the Chain (continued)

Then:

D .

Finally:

E .

Complete Structure

Thus:

$$n \text{ odd}$$

$$\Rightarrow n = 2k + 1$$

$$\Rightarrow n^2 = 4k(k + 1) + 1$$

$$\Rightarrow k(k + 1) \text{ even}$$

$$\Rightarrow 8 \mid 4k(k + 1)$$

$$\Rightarrow \boxed{n^2 \equiv 1 \pmod{8}}.$$

This is a genuine multi-stage proof chain.

A Chain Can Look Convincing and Still Fail

Consider:

$$x^2 = 9$$

$$\Rightarrow x = 3$$

$$\Rightarrow x > 0.$$

What is wrong?

The first implication is false.

From

$$x^2 = 9$$

we obtain

$$x = 3 \quad \text{or} \quad x = -3.$$

Therefore the chain fails at its first link.

Example 16

Consider:

$$ab = ac$$

$$\Rightarrow b = c.$$

Is this always valid?

No.

If

$$a = 0,$$

then

$$ab = ac$$

holds for every b, c .

For example:

Example 16 (continued)

$$a = 0, \quad b = 2, \quad c = 5.$$

Then:

$$ab = ac = 0$$

but

$$b \neq c.$$

The Missing Condition

The cancellation

$$ab = ac \Rightarrow b = c$$

is valid if

$$a \neq 0.$$

Because then we may divide both sides by a .

This illustrates a general principle:

Every algebraic operation has conditions under which it is valid.

Another Invalid Step

Suppose:

$$\sin A = \sin B.$$

Can we conclude:

$$A = B?$$

Not in general.

The sine function takes the same value at different angles.

Thus:

$$\sin A = \sin B \not\Rightarrow A = B$$

without additional restrictions.

TMUA Example 1

Suppose:

$$A \Rightarrow B,$$

$$B \Rightarrow C,$$

and A is true.

Which statement must be true?

(A) $\neg B$

(B) C

(C) $\neg C$

(D) none of these

TMUA Example 1 (continued)

Since

$$A \Rightarrow B \Rightarrow C,$$

the answer is

(B).

TMUA Example 2

Suppose:

$$A \Rightarrow B,$$

$$C \Rightarrow B,$$

and B is true.

Which must follow?

(A) A

(B) C

(C) $A \wedge C$

(D) none of these

TMUA Example 2 (continued)

The answer is:

(D).

Knowing the conclusion does not tell us which sufficient condition produced it.

TMUA Example 3

Suppose:

$$A \Rightarrow B,$$

$$B \Rightarrow C,$$

and C is false.

What can we conclude?

From

$$B \Rightarrow C$$

and $\neg C$,

the contrapositive gives:

$$\neg B.$$

Then from

TMUA Example 3 (continued)

$$A \Rightarrow B$$

and $\neg B$,

we obtain:

$$\neg A.$$

Thus:

$$\boxed{\neg A \text{ and } \neg B}.$$

TMUA Example 4

Suppose:

$$P \Rightarrow Q,$$

$$R \Rightarrow P,$$

$$Q \Rightarrow S.$$

If R is true, what must follow?

Start with:

$$R.$$

Then:

$$R \Rightarrow P \Rightarrow Q \Rightarrow S.$$

Therefore:

TMUA Example 4 (continued)

$$\boxed{S}.$$

TMUA Example 5

Suppose:

$$A \Rightarrow B,$$

$$C \Rightarrow D,$$

$$B \Rightarrow C.$$

Which ordering reveals the complete chain?

$$A \Rightarrow B \Rightarrow C \Rightarrow D.$$

The statements may be presented in an inconvenient order.
Your job is to reconstruct their dependency.

TMUA Example 6

The first few values of a sequence are

$$2, 6, 12, 20, 30.$$

A student notices:

$$2 = 1 \cdot 2,$$

$$6 = 2 \cdot 3,$$

$$12 = 3 \cdot 4,$$

$$20 = 4 \cdot 5.$$

A natural conjecture is:

$$a_n = n(n+1).$$

TMUA Example 6 (continued)

But what has been established?
Only that the formula matches the observed cases.

Many Rules Can Fit Finite Data

A finite list of values does not normally determine a unique infinite sequence.
The values

2, 6, 12, 20, 30

are consistent with

$$a_n = n(n + 1),$$

but infinitely many more complicated rules could agree with those first five terms and differ later.
Therefore:

pattern recognition produces a conjecture, not certainty.

TMUA Example 7

Suppose calculations suggest:

$$n^2 + n + 2$$

is even for

$$n = 1, 2, 3, 4, 5.$$

Which observation proves the conjecture for all integers?

$$n^2 + n + 2 = n(n + 1) + 2.$$

Since

$$n(n + 1)$$

is always even and 2 is even,

$n^2 + n + 2$ is always even.

TMUA Example 7 (continued)

The structural argument completes the conjecture.

What Makes a Chain Difficult?

The individual steps may all be elementary.

Difficulty can arise because:

- the useful intermediate statement is hidden;
- several facts must be combined;
- some information is irrelevant;
- a result from one stage becomes input to another;
- the direction of implication must be tracked carefully;
- the proof requires switching representation.

Switching Representation

A statement may become easier when rewritten.

For example:

n is even

can become:

$$n = 2k.$$

A divisibility statement

$$d \mid n$$

can become:

$$n = dk.$$

An algebraic target may suggest:

Switching Representation (continued)

factorisation,

expansion,

or

substitution.

Changing representation often creates the next link in the chain.

Example 17

Suppose:

$$x + y = 5, \quad xy = 6.$$

Find:

$$x^3 + y^3.$$

Use the identity:

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

Therefore:

$$x^3 + y^3 = 5^3 - 3(6)(5).$$

Hence:

$$125 - 90 = 35.$$

Example 17 (continued)

So:

$$x^3 + y^3 = 35.$$

Why the Identity Was Useful

The target is:

$$x^3 + y^3.$$

The given information is:

$$x + y \quad \text{and} \quad xy.$$

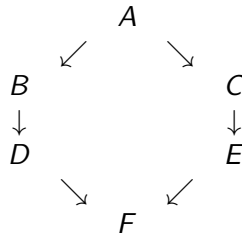
Therefore we seek an identity connecting exactly these quantities.
That gives:

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

This is backward planning followed by forward execution.

From Chain to Network

Real mathematical reasoning is sometimes better represented as:



Different consequences may be developed separately and later combined.

Reading the Network

From A , suppose we obtain both B and C .

From B , obtain D .

From C , obtain E .

Finally, D and E together establish F .

The proof structure is:

$$A \Rightarrow (B \wedge C)$$

$$\Rightarrow (D \wedge E)$$

$$\Rightarrow F.$$

This is more sophisticated than a single straight chain.

A Practical Reasoning Algorithm

When faced with a difficult Paper 2 argument:

- ① Identify exactly what is given.
- ② Identify exactly what must be shown.
- ③ Extract immediate consequences of the hypotheses.
- ④ Work backward from the target.
- ⑤ Look for statements that connect the two sides.
- ⑥ Check every implication direction.
- ⑦ Remove irrelevant information.
- ⑧ Test suspicious steps with counterexamples.

Step 1: Translate the Problem

Before calculating, identify:

Hypotheses

and

Conclusion.

Write mentally:

Given $\longrightarrow$ Wanted.

This prevents aimless algebra.

Step 2: Expand the Given Information

Ask:

What does each hypothesis really mean?

For example:

$$n \text{ even} \Rightarrow n = 2k.$$

$$a \mid b \Rightarrow b = ak.$$

$$x^2 = 9 \Rightarrow x = \pm 3.$$

The useful consequence is often hidden inside a definition.

Step 3: Inspect the Target

Ask:

What would be enough to prove this?

If the target is:

$$x^2 + y^2,$$

perhaps

$$(x + y)^2$$

and xy are relevant.

If the target is divisibility by 4,

perhaps we should aim to write the expression as:

$$4k.$$

The form of the target often suggests the route.

Step 4: Meet in the Middle

Ideally:

$$\text{Given} \Rightarrow P \Rightarrow Q$$

while backward planning suggests:

$$R \Rightarrow S \Rightarrow \text{Target}.$$

The key discovery is a connection:

$$Q \Rightarrow R.$$

Then the whole proof becomes:

$$\boxed{\text{Given} \Rightarrow P \Rightarrow Q \Rightarrow R \Rightarrow S \Rightarrow \text{Target}.}$$

Challenge 1

Suppose:

$$A \Rightarrow B, \quad B \Rightarrow C, \quad C \Rightarrow D.$$

If A is true, what must be true?

Challenge 2

Suppose:

$$A \Rightarrow B$$

and

$$B \Rightarrow C.$$

If C is false, what follows?

Challenge 2 (continued)

$$\boxed{\neg A \text{ and } \neg B}.$$

Challenge 3

Arrange the statements:

(A) $n^2 = 4k^2$,

(B) $n = 2k$,

(C) n^2 is even,

(D) n is even.

Challenge 4

The first few values of

$$n^2 - n + 2$$

are:

2, 4, 8, 14, 22.

They are all even.

Prove the pattern.

Challenge 5

Suppose:

$$x + \frac{1}{x} = 4.$$

Find:

$$x^2 + \frac{1}{x^2}.$$

Challenge 6

Suppose:

$$a + b = 8, \quad ab = 12.$$

Find:

$$a^2 + b^2.$$

Challenge 7

A student argues:

$$x^2 > 4 \Rightarrow x > 2.$$

Find the flaw.

Challenge 7 (continued)

$$x > 2 \text{ or } x < -2.$$

Challenge 8

Suppose:

$$P \Rightarrow Q,$$

$$Q \Rightarrow R,$$

$$S \Rightarrow P.$$

What follows from S ?

Common Error 1: Reversing a Link

Given:

$$A \Rightarrow B,$$

a student uses:

$$B \Rightarrow A.$$

Unless the converse is independently established, the proof is invalid.
Every arrow in the chain must point in a justified direction.

Common Error 2: A True but Useless Step

Suppose we need to prove:

$$x^2 > 9$$

from

$$x > 3.$$

A student writes:

$$x > 3 \Rightarrow x > 0.$$

This is true.

But by itself it is not sufficient to reach the target.

A proof step should not only be correct.

It should move the argument toward the conclusion.

Common Error 3: Guessing Is Not Justifying

Checking:

$$n = 1, 2, 3, 4, 5$$

may suggest a conjecture.

It does not establish a statement for every integer.

You still need:

a general argument.

Common Error 4: Hidden Division by Zero

From:

$$ab = ac$$

a student cancels a .

This requires:

$$a \neq 0.$$

If the proof has not established that condition, the step may be invalid.

Always ask:

Under what conditions is this operation legal?

Common Error 5: Ignoring Alternatives

From:

$$x^2 = 16,$$

do not write only:

$$x = 4.$$

The correct deduction is:

$$x = \pm 4.$$

Dropping a legitimate branch can destroy an otherwise correct chain of reasoning.

Common Error 6: Assuming Every Fact Is Needed

More information does not necessarily mean every fact belongs in the proof.
Some information may be:

irrelevant

to the required conclusion.
A good solution identifies the shortest valid dependency chain.

Think in Dependencies

Instead of seeing a proof as a paragraph of algebra, see it as a dependency structure.
Ask:

What depends on what?

Which statement must come first?

Which fact unlocks the next step?

Which information is unnecessary?

This perspective makes long arguments much easier to control.

Separate Discovery from Verification

When solving:

Discovery

means finding a possible route.
But then comes:

Verification.

Check:

Is every implication valid?

Are all conditions satisfied?

Have any cases been lost?

Separate Discovery from Verification (continued)

Did I accidentally use a converse?

Reasoning Is More Important Than Length

A long solution is not automatically sophisticated.

A short solution is not automatically simple.

The quality of an argument depends on:

validity

relevance

structure

and

clarity of dependency.

The goal is not to write more steps.

The goal is to find the right steps.

Build a Mental Graph

As you read a problem, mentally create:

Given facts

connected to

immediate consequences

connected to

intermediate results

connected to

target.

This transforms a complicated problem into a navigable logical network.

Skill 1: Deduce Implications

From:

$$A \Rightarrow B$$

and

$$B \Rightarrow C,$$

deduce:

$$\boxed{A \Rightarrow C}.$$

More generally:

$$A \Rightarrow B \Rightarrow C \Rightarrow D$$

gives:

$$\boxed{A \Rightarrow D}.$$

Skill 2: Conjecture and Justify

The correct workflow is:

Small Cases



Pattern



Conjecture



Skill 2: Conjecture and Justify (continued)

Test



General Justification.

Skill 3: Rearrange Proofs

To reconstruct a proof:

Find the hypothesis



Identify immediate consequences



Follow dependencies



Skill 3: Rearrange Proofs (continued)

Reach the required conclusion .

Do not order statements merely by how they look.
Order them by logical dependence.

Skill 4: Sophisticated Reasoning

Longer problems may require:

forward reasoning

backward planning

branching

combining information

changing representation

and

testing suspicious steps .

These skills work together rather than separately.

Self-Check

Before moving on, make sure you can:

- ① Deduce consequences from given statements.
- ② Combine several implications.
- ③ Work forward from a hypothesis.
- ④ Plan backward from a target.
- ⑤ Form conjectures from small cases.
- ⑥ Distinguish conjecture from proof.
- ⑦ Justify a conjecture generally.
- ⑧ Rearrange scrambled proof steps.
- ⑨ Identify missing steps.
- ⑩ Detect invalid implications.
- ⑪ Separate relevant from irrelevant information.
- ⑫ Follow longer chains of mathematical reasoning.

The Central Habit

When solving a difficult reasoning problem, repeatedly ask:

What do I know?

What follows?

What do I need?

What would be sufficient?

and finally:

Does every step really follow?

These questions turn complicated arguments into manageable logical structures.

Coming in Part 10

Identifying Errors in Mathematical Proofs

A proof can look convincing and still contain one fatal step.
In Part 10 we will learn how to audit an argument line by line.

We will study:

Invalid Cancellation

Division by Zero

Lost or Introduced Solutions

Coming in Part 10 (continued)

False Converses

Trigonometric Non-Uniqueness

Hidden Assumptions

and

Finding the First Invalid Step.

Start With What You Know. Find What Must Follow. Build the Chain Carefully.

Mathematics Elevate Academy

Rishabh Kumar

TMUA Paper 2 Masterclass

Part 10: Identifying Errors in Mathematical Proofs
Finding the First Invalid Step

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A Proof Can Look Convincing and Still Be Wrong

Consider an argument containing twenty lines.
Suppose:

Lines 1–12 are correct,

but

Line 13 contains an invalid deduction.

Even if lines 14–20 are algebraically correct consequences of line 13, the argument has failed.

Key Idea

A mathematical proof is valid only when every required step is logically and mathematically justified.

The Central Question

When reading a purported proof, do not ask only:

Does the final answer look correct?

Ask:

Does each statement follow from what came before?

The central skill of this lesson is therefore:

Locate the first invalid step.

What Paper 2 Is Testing

In error-identification problems you may need to recognise:

- an unjustified implication;
- an invalid converse;
- illegal cancellation;
- hidden division by zero;
- a lost solution or extra solution;
- an ignored domain restriction;
- a trigonometric non-uniqueness;
- an unjustified assumption;
- a counterexample to a claimed step.

The mistakes can be mathematically elementary.

Recognising them quickly may not be.

A Proof-Auditing Mindset

For every line, mentally ask:

Why is this true?

Then:

What condition is being used?

Then:

Is the implication reversible?

And finally:

Can I break this step with a counterexample?

Why the First Error Matters

Suppose:

$$A \Rightarrow B$$

is valid,

$$B \Rightarrow C$$

is valid,
but

$$C \Rightarrow D$$

is invalid.

Then everything after D may become unreliable.

So the most useful diagnosis is often:

$$C \not\Rightarrow D.$$

Why the First Error Matters (continued)

Do not merely say:

"The proof is wrong."

Identify precisely where and why.

A Three-Step Audit

For each transition

$$P \Rightarrow Q,$$

ask:

- ❶ Is the claimed implication actually true?
- ❷ Are all required conditions satisfied?
- ❸ Has the operation changed the set of possible values?

If any answer is problematic, investigate that step.

Cancellation Looks Innocent

Suppose:

$$ab = ac.$$

A student writes:

$$b = c.$$

Is this always valid?

No.

To cancel a , we are effectively dividing both sides by a .

That requires:

$$\boxed{a \neq 0}.$$

A Counterexample

Take:

$$a = 0, \quad b = 2, \quad c = 7.$$

Then:

$$ab = 0$$

and

$$ac = 0.$$

So:

$$ab = ac.$$

But:

$$2 \neq 7.$$

A Counterexample (continued)

Therefore:

$$ab = ac \not\Rightarrow b = c$$

without the condition $a \neq 0$.

The Correct Statement

If

$$ab = ac$$

and

$$a \neq 0,$$

then dividing by a gives:

$$b = c.$$

Thus the valid implication is:

$$ab = ac \text{ and } a \neq 0 \Rightarrow b = c.$$

TMUA Thinking

When something is cancelled, ask what has actually been divided out and whether it could be zero.

Can We Prove $1 = 2$?

Consider the following argument.

Suppose:

$$a = b.$$

Then:

$$a^2 = ab.$$

Subtract b^2 :

$$a^2 - b^2 = ab - b^2.$$

Factor:

$$(a - b)(a + b) = b(a - b).$$

Cancel $a - b$:

Can We Prove $1 = 2$? (continued)

$$a + b = b.$$

Since $a = b$:

$$2b = b.$$

Therefore:

$$\boxed{2 = 1}.$$

Where is the error?

Find the First Invalid Step

We began with:

$$a = b.$$

Therefore:

$$a - b = 0.$$

The argument then cancels:

$$(a - b)(a + b) = b(a - b).$$

But cancelling $a - b$ means dividing by:

$$a - b = 0.$$

Therefore the invalid step is:

division by zero.

Why the Earlier Lines Were Fine

Notice carefully:

$$a = b$$

is allowed.

Then:

$$a^2 = ab$$

is valid.

Then:

$$a^2 - b^2 = ab - b^2$$

is valid.

Then:

$$(a - b)(a + b) = b(a - b)$$

Why the Earlier Lines Were Fine (continued)

is also valid.

The proof does not fail until the cancellation.

Key Idea

Do not blame the entire argument. Locate the exact transition where validity is lost.

The Converse Trap

Suppose we know:

$$4 \mid n \Rightarrow n \text{ is even.}$$

A student argues:

$$n \text{ is even} \Rightarrow 4 \mid n.$$

This reverses the implication.

It is the converse.

The original statement being true does not make its converse true.

Break the Converse

Take:

$$n = 6.$$

Then:

$$n \text{ is even}$$

is true.

But:

$$4 \nmid 6.$$

Therefore:

$$n \text{ even} \not\Rightarrow 4 \mid n.$$

A single counterexample exposes the error.

Another Converse Error

True statement:

$$x > 5 \Rightarrow x > 2.$$

Invalid reversal:

$$x > 2 \Rightarrow x > 5.$$

Counterexample:

$$x = 3.$$

Thus the direction of implication matters.

$A \Rightarrow B$ does not generally give $B \Rightarrow A$.

A Logical Error

Suppose:

$$A \Rightarrow B.$$

We are told:

B is true.

Can we conclude:

A is true?

No.

There may be many different reasons why B is true.

Example

Suppose:

$$4 \mid n \Rightarrow n \text{ is even.}$$

Now suppose:

$$n \text{ is even.}$$

It does not follow that:

$$4 \mid n.$$

Again:

$$n = 6$$

is enough to disprove the deduction.

Example (continued)

Common Mistake

Knowing a consequence does not generally allow us to reconstruct the condition that produced it.

Another Logical Trap

Suppose:

$$A \Rightarrow B.$$

We discover:

A is false.

Can we conclude:

B is false?

No.

The implication tells us what happens when A is true.

It does not say that A is the only way B can occur.

Example

Suppose:

$$4 \mid n \Rightarrow n \text{ is even.}$$

Now take:

$$4 \nmid n.$$

Can we conclude n is odd?

No.

For example:

$$n = 6.$$

Then:

$$4 \nmid 6$$

but 6 is still even.

The Valid Reverse-Looking Argument

From:

$$A \Rightarrow B,$$

we may validly infer the contrapositive:

$$\neg B \Rightarrow \neg A.$$

This is very different from:

$$B \Rightarrow A$$

or

$$\neg A \Rightarrow \neg B.$$

Keep the Four Forms Separate

Statement	$A \Rightarrow B$
Converse	$B \Rightarrow A$
Contrapositive	$\neg B \Rightarrow \neg A$
Negate both without swapping	$\neg A \Rightarrow \neg B$

Only the original statement and its contrapositive are automatically logically equivalent.

Squaring Can Lose Information

Suppose:

$$x = 3.$$

Then certainly:

$$x^2 = 9.$$

So:

$$x = 3 \Rightarrow x^2 = 9.$$

But can we reverse this?

$$x^2 = 9 \Rightarrow x = 3?$$

No.

The Missing Possibility

From:

$$x^2 = 9$$

we obtain:

$$x = 3 \text{ or } x = -3.$$

Thus:

$$x^2 = 9 \not\Rightarrow x = 3.$$

Squaring has removed sign information.

TMUA Thinking

Whenever an operation is many-to-one, reversing it requires special care.

A General Warning

From:

$$x = y$$

we may conclude:

$$x^2 = y^2.$$

But from:

$$x^2 = y^2$$

we obtain:

$$x = y \quad \text{or} \quad x = -y.$$

Equivalently:

$$x^2 - y^2 = 0$$

A General Warning (continued)

gives

$$(x - y)(x + y) = 0.$$

Therefore:

$$\boxed{x = \pm y}.$$

The Principal Square Root

A common error is:

$$\sqrt{x^2} = x.$$

This is not true for every real x .
For example, if:

$$x = -5,$$

then:

$$\sqrt{x^2} = \sqrt{25} = 5,$$

not -5 .

The correct identity is:

$$\boxed{\sqrt{x^2} = |x|}.$$

Why the Absolute Value Appears

The principal square root is non-negative.
Therefore:

$$\sqrt{x^2} \geq 0.$$

If $x \geq 0$, then:

$$\sqrt{x^2} = x.$$

If $x < 0$, then:

$$\sqrt{x^2} = -x.$$

Both cases are summarized by:

$$\boxed{\sqrt{x^2} = |x|}.$$

Squaring May Introduce Extra Solutions

Consider:

$$x = -2.$$

Squaring gives:

$$x^2 = 4.$$

But solving:

$$x^2 = 4$$

gives:

$$x = \pm 2.$$

The value $x = 2$ was not a solution of the original equation.
Thus squaring can enlarge the solution set.

Example

Solve:

$$\sqrt{x} = x - 2.$$

Squaring gives:

$$x = (x - 2)^2.$$

So:

$$x = x^2 - 4x + 4,$$

hence:

$$x^2 - 5x + 4 = 0.$$

Therefore:

$$(x - 1)(x - 4) = 0.$$

Example (continued)

Candidate solutions:

$$x = 1, \quad x = 4.$$

Are both valid?

Check Against the Original

For $x = 1$:

$$\sqrt{1} = 1,$$

but:

$$1 - 2 = -1.$$

So $x = 1$ is invalid.

For $x = 4$:

$$\sqrt{4} = 2$$

and

$$4 - 2 = 2.$$

Therefore:

Check Against the Original (continued)

$$\boxed{x = 4}.$$

Key Idea

After a non-reversible operation such as squaring, check candidate solutions in the original equation.

Operations Can Change Equivalence

Suppose:

$$x = 1.$$

Multiplying both sides by $x - 2$ gives:

$$x(x - 2) = x - 2.$$

That implication is valid.

But reversing the resulting equation may introduce another solution.

Solve:

$$x(x - 2) = x - 2.$$

The New Solution

Rearrange:

$$x(x - 2) - (x - 2) = 0.$$

Factor:

$$(x - 2)(x - 1) = 0.$$

Hence:

$$x = 1 \quad \text{or} \quad x = 2.$$

But the original statement was:

$$x = 1.$$

Thus multiplication by an expression that can equal zero may destroy reversibility.

Division Can Remove a Valid Case

Solve:

$$x(x - 3) = 0.$$

A student divides both sides by x :

$$x - 3 = 0.$$

Hence:

$$x = 3.$$

What solution has been lost?

The Lost Case

The original equation:

$$x(x - 3) = 0$$

has solutions:

$$x = 0$$

or

$$x = 3.$$

Dividing by x silently assumes:

$$x \neq 0.$$

Therefore the solution $x = 0$ disappears.

The Lost Case (continued)

Common Mistake

Before dividing by a variable expression, consider separately the case in which that expression equals zero.

Multiplying Inequalities

Suppose:

$$a < b.$$

A student multiplies by c and writes:

$$ac < bc.$$

Is this always correct?

No.

The result depends on the sign of c .

Three Cases

If:

$$c > 0,$$

then:

$$a < b \Rightarrow ac < bc.$$

If:

$$c < 0,$$

then the inequality reverses:

$$a < b \Rightarrow ac > bc.$$

If:

$$c = 0,$$

Three Cases (continued)

then:

$$ac = bc = 0.$$

Therefore the sign of the multiplier matters.

Example

Start with:

$$2 < 5.$$

Multiply by -3 :

$$-6 > -15.$$

Not:

$$-6 < -15.$$

Thus multiplying or dividing an inequality by a negative quantity reverses the inequality.

Does Squaring Preserve Order?

Suppose:

$$x < y.$$

Can we conclude:

$$x^2 < y^2?$$

No.

Take:

$$x = -3, \quad y = 2.$$

Then:

$$-3 < 2$$

but:

Does Squaring Preserve Order? (continued)

$$9 > 4.$$

Thus:

$$x < y \not\Rightarrow x^2 < y^2.$$

When Squaring Does Preserve Order

If:

$$0 \leq x < y,$$

then:

$$x^2 < y^2.$$

So the missing issue is the domain.

Key Idea

An algebraic step may be valid on one domain and invalid on another.

A Crucial Trigonometric Error

Suppose:

$$\sin A = \sin B.$$

A student concludes:

$$A = B.$$

This is not valid in general.

Why?

Because sine is not one-to-one over all real angles.

A Simple Counterexample

Consider:

$$A = 30^\circ, \quad B = 150^\circ.$$

Then:

$$\sin 30^\circ = \frac{1}{2}$$

and:

$$\sin 150^\circ = \frac{1}{2}.$$

Therefore:

$$\sin A = \sin B$$

but:

A Simple Counterexample (continued)

$$A \neq B.$$

So:

$$\sin A = \sin B \not\Rightarrow A = B.$$

Why This Happens

The sine graph repeats:

$$\sin(\theta + 360^\circ) = \sin \theta.$$

It also has the symmetry:

$$\sin(180^\circ - \theta) = \sin \theta.$$

Therefore many different angles can have the same sine.

Equality of function values does not imply equality of inputs unless the function is one-to-one on the relevant domain.

The General Principle

Suppose:

$$f(x) = f(y).$$

Can we conclude:

$$x = y?$$

Only if f is injective on the relevant domain.

For example:

$$x^2 = y^2$$

does not imply $x = y$,
because $f(x) = x^2$ is not one-to-one on all real numbers.

Likewise:

$$\sin A = \sin B$$

does not generally imply $A = B$.

Equal Outputs Need Not Mean Equal Inputs

Consider:

$$f(x) = x^2.$$

Then:

$$f(3) = 9$$

and:

$$f(-3) = 9.$$

Therefore:

$$f(3) = f(-3)$$

although:

$$3 \neq -3.$$

Equal Outputs Need Not Mean Equal Inputs (continued)

So the deduction

$$f(a) = f(b) \Rightarrow a = b$$

requires additional information about f .

Expressions Have Domains

Consider:

$$\frac{x^2 - 1}{x - 1}.$$

Factor:

$$\frac{(x - 1)(x + 1)}{x - 1}.$$

Cancel:

$$x + 1.$$

Does this mean:

$$\frac{x^2 - 1}{x - 1} = x + 1$$

for every real x ?

No.

The Missing Domain Condition

The original expression:

$$\frac{x^2 - 1}{x - 1}$$

is undefined at:

$$x = 1.$$

Therefore the correct statement is:

$$\boxed{\frac{x^2 - 1}{x - 1} = x + 1 \text{ for } x \neq 1.}$$

Cancellation simplifies an expression on its domain.

It does not automatically restore excluded values.

Logarithmic Restrictions

Consider:

$$\log(x^2).$$

A student writes:

$$\log(x^2) = 2 \log x.$$

For real logarithms, the right-hand side requires:

$$x > 0.$$

But the left-hand side exists whenever:

$$x \neq 0.$$

For example, $x = -2$ makes the left side meaningful but $\log x$ is not real.
Domain conditions matter.

Absolute Value

Suppose:

$$|x| = 3.$$

A student concludes:

$$x = 3.$$

This loses a valid possibility.
The correct conclusion is:

$$x = 3 \text{ or } x = -3.$$

Whenever absolute values appear, consider the relevant cases.

Another Absolute Value Trap

Is:

$$|x + y| = |x| + |y|$$

always true?

No.

Take:

$$x = 1, \quad y = -1.$$

Then:

$$|x + y| = 0$$

but:

$$|x| + |y| = 2.$$

The valid general relationship is:

Another Absolute Value Trap (continued)

$$\boxed{|x + y| \leq |x| + |y|}.$$

Changing “For All” into “For Some”

Suppose a student verifies:

$$P(1), \quad P(2), \quad P(3).$$

Can they conclude:

$$P(n) \text{ for every positive integer } n?$$

No.

Finite evidence can suggest a conjecture.

It does not generally prove a universal statement.

A Counterexample Is Enough

To disprove:

$\text{For every } x, P(x)$

we need only one value for which:

$$P(x)$$

is false.

So the negation has the structure:

$\text{There exists } x \text{ for which } P(x) \text{ is false.}$

This makes counterexamples particularly powerful when auditing universal claims.

Do Not Swap Quantifiers

Compare:

For every x , there exists y

with:

There exists y such that for every x .

These need not mean the same thing.

For example:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } y > x$$

is true.

But:

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y > x$$

is false.

A Contradiction Must Actually Follow

In proof by contradiction we assume the negation of the desired statement.

Then we must derive an actual contradiction using valid steps.

It is not enough to derive:

something surprising

or:

something inconvenient.

The argument must force incompatible statements.

The Structure Must Be Valid

To prove A , suppose:

$$\neg A.$$

Then derive:

$$B$$

and:

$$\neg B.$$

Therefore the assumption $\neg A$ cannot be maintained.

Hence:

$$A.$$

But if one step on the route to B or $\neg B$ is invalid, the contradiction proves nothing.

Try to Break the Claim

Suppose a proof uses:

$$P \Rightarrow Q.$$

If you suspect this implication, search for:

P true and Q false.

Finding such an example immediately proves that the step is invalid.

Example

Claim:

$$x < y \Rightarrow x^2 < y^2.$$

To break it we need:

$$x < y$$

true,
but:

$$x^2 < y^2$$

false.

Choose:

$$x = -2, \quad y = 1.$$

Then:

Example (continued)

$$-2 < 1$$

but:

$$4 \not< 1.$$

Therefore the implication fails.

Strategic Counterexample Search

Useful values to test include:

0

because of division and cancellation,

$1, -1$

because powers simplify,

negative values

for inequalities and squares,

boundary values

for domains,
and

Strategic Counterexample Search (continued)

simple special angles

for trigonometric claims.

Audit This Proof

Claim:

Every positive real number equals 1.

Let $x > 0$.

Suppose:

$$x = 1.$$

Then:

$$x^2 = x.$$

So:

$$x^2 - x = 0.$$

Hence:

Audit This Proof (continued)

$$x(x - 1) = 0.$$

Cancel $x - 1$:

$$x = 0.$$

Something has clearly gone wrong.
Where?

Audit Carefully

There are actually multiple conceptual problems.
First, the argument began by assuming:

$$x = 1.$$

That cannot establish a claim about every positive real x .
Second, if $x = 1$, then:

$$x - 1 = 0.$$

So cancelling $x - 1$ is illegal.
This illustrates an important lesson:

A purported proof may contain more than one error.

But identify the earliest one relevant to the claimed argument.

Audit This Argument

Claim:

$$x^2 > 4 \Rightarrow x > 2.$$

A student writes:

$$x^2 > 4.$$

Taking square roots:

$$x > 2.$$

Where is the error?

Audit This Argument (continued)

$$|x| > 2.$$

Therefore:

$$x > 2 \text{ or } x < -2.$$

Audit This Argument

Claim:

$$\sin A = \sin B \Rightarrow A = B.$$

Proof:

$$\sin A = \sin B.$$

Apply $\sin^{-1}$ to both sides:

$$\sin^{-1}(\sin A) = \sin^{-1}(\sin B).$$

Therefore:

$$A = B.$$

Where is the problem?

The Inverse Is Restricted

The expression

$$\sin^{-1}(\sin \theta)$$

does not equal θ for every real angle θ .

The inverse sine returns a value in its principal range.

For example:

$$\sin 150^\circ = \frac{1}{2},$$

but:

$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ.$$

Thus:

$$\sin^{-1}(\sin 150^\circ) \neq 150^\circ.$$

Audit This Argument

Claim:

$$a^2 = b^2 \Rightarrow a = b.$$

Proof:

$$a^2 = b^2.$$

Taking square roots:

$$a = b.$$

The error is:

$$\sqrt{a^2} = |a|$$

and

$$\sqrt{b^2} = |b|.$$

Audit This Argument (continued)

Therefore we obtain:

$$|a| = |b|,$$

which means:

$$a = b \text{ or } a = -b.$$

Audit This Argument

Suppose:

$$x(x - 5) = 0.$$

A student divides by x :

$$x - 5 = 0.$$

Therefore:

$$x = 5.$$

The step is valid only if:

$$x \neq 0.$$

But $x = 0$ is itself a solution of the original equation.
Therefore division has removed a valid case.

Audit This Argument

Claim:

$$a < b \Rightarrow a^2 < b^2.$$

Proof:

$$a < b.$$

Multiply both sides by a :

$$a^2 < ab.$$

Multiply the original inequality by b :

$$ab < b^2.$$

Therefore:

$$a^2 < ab < b^2.$$

Audit This Argument (continued)

Hence:

$$a^2 < b^2.$$

Where is the hidden problem?

The Signs Were Never Established

From:

$$a < b$$

we do not know that a and b are positive.

Multiplying an inequality by a or b may reverse its direction if the multiplier is negative.

For example:

$$a = -3, \quad b = -2.$$

Then:

$$a < b$$

but:

$$9 > 4.$$

So the argument silently assumes sign information it does not have.

Audit This Argument

Claim:

$$\frac{x^2 - 4}{x - 2} = x + 2.$$

Proof:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Is the statement correct for every real x ?

No.

At:

$$x = 2,$$

the original expression is undefined.

Therefore:

Audit This Argument (continued)

$$\frac{x^2 - 4}{x - 2} = x + 2 \quad \text{only for } x \neq 2.$$

Audit This Argument

Claim:

All prime numbers are odd.

Argument:

3, 5, 7, 11, 13, 17

are all prime and all odd.

Therefore all prime numbers are odd.

What is wrong?

The examples do not establish a universal statement.

Indeed:

2

is prime and even.

One counterexample destroys the claim.

TMUA Example 1

A student argues:

$$xy = xz$$

therefore:

$$y = z.$$

Which additional condition would make the deduction valid?

(A) $x = 0$

(B) $x \neq 0$

(C) $y \neq 0$

(D) $z \neq 0$

TMUA Example 1 (continued)

The answer is:

(B).

TMUA Example 2

Which deduction is not valid for all real x, y ?

(A) $x = y \Rightarrow x + 1 = y + 1$

(B) $x = y \Rightarrow x^2 = y^2$

(C) $x^2 = y^2 \Rightarrow x = y$

(D) $x = y \Rightarrow 3x = 3y$

The invalid deduction is:

(C).

TMUA Example 3

A student knows:

$$n^2 \text{ is even.}$$

They claim:

$$n \text{ is divisible by 4.}$$

Which value gives a counterexample?

Take:

$$n = 2.$$

Then:

$$n^2 = 4$$

is even,

but:

TMUA Example 3 (continued)

$$4 \nmid 2.$$

Therefore the claimed implication is false.

TMUA Example 4

Suppose:

$$x < y.$$

Which additional condition guarantees:

$$x^2 < y^2?$$

One sufficient condition is:

$$0 \leq x < y.$$

On the non-negative real numbers, the squaring function is strictly increasing.
Without such a restriction, the implication can fail.

TMUA Example 5

A student solves:

$$\sqrt{x+2} = x$$

by squaring:

$$x+2 = x^2.$$

They obtain candidate values.
What must they do next?

Check every candidate in the original equation.

Why?

Because squaring is not reversible without additional conditions and may introduce extraneous solutions.

A More Subtle Audit

Consider:

$$x = y.$$

Then:

$$x^2 = xy.$$

Hence:

$$x^2 - y^2 = xy - y^2.$$

Therefore:

$$(x-y)(x+y) = y(x-y).$$

Hence:

$$x+y = y.$$

A More Subtle Audit (continued)

Therefore:

$$2y = y.$$

Hence:

$$2 = 1.$$

Which is the first invalid line?

Trace the Argument

These are valid:

$$x = y \Rightarrow x^2 = xy,$$

$$x^2 = xy \Rightarrow x^2 - y^2 = xy - y^2,$$

and:

$$x^2 - y^2 = xy - y^2 \Rightarrow (x - y)(x + y) = y(x - y).$$

The next step cancels $x - y$.

But:

$$x = y \Rightarrow x - y = 0.$$

Therefore:

the cancellation is the first invalid step.

Do More Than Find the Error

A deeper skill is asking:

Can the argument be repaired?

For example:

$$ab = ac \Rightarrow b = c$$

is false in general.

But adding:

$$a \neq 0$$

repairs it:

$$ab = ac, a \neq 0 \Rightarrow b = c.$$

Repairing a Squaring Argument

False in general:

$$x < y \Rightarrow x^2 < y^2.$$

One repair is:

$$0 \leq x < y \Rightarrow x^2 < y^2.$$

The important question becomes:

What missing condition makes the step valid?

This is often more informative than merely labelling it false.

Repairing the Sine Statement

False in general:

$$\sin A = \sin B \Rightarrow A = B.$$

But if A and B are restricted to an interval on which sine is one-to-one, the deduction can become valid. For example, if:

$$-\frac{\pi}{2} \leq A, B \leq \frac{\pi}{2},$$

then:

$$\sin A = \sin B \Rightarrow A = B.$$

The missing ingredient was a domain restriction.

The Proof Audit Checklist

When presented with a purported proof:

- ➊ Identify the claim.
- ➋ Identify the assumptions.
- ➌ Read one transition at a time.
- ➍ Check the direction of every implication.
- ➎ Inspect every cancellation or division.
- ➏ Check signs before manipulating inequalities.
- ➐ Check domains.
- ➑ Watch non-reversible operations.
- ➒ Look for lost or introduced cases.
- ➓ Search for a counterexample to suspicious steps.

Red Flag: Cancellation

Whenever you see:

$$AB = AC$$

becoming:

$$B = C,$$

ask:

$$\boxed{A \neq 0?}$$

Whenever you see:

$$\frac{AB}{A} = B,$$

ask:

$$\boxed{A \neq 0?}$$

Red Flag: Cancellation (continued)

Cancellation always carries a condition.

Red Flag: Squaring

Whenever you see:

$$P \Rightarrow P^2,$$

ask whether the reverse direction is being used later.

Squaring can:

lose sign information

and

introduce extra solutions.

Check reversibility carefully.

Red Flag: Square Roots

Whenever you see:

$$\sqrt{x^2},$$

remember:

$$\sqrt{x^2} = |x|.$$

Whenever you see:

$$x^2 = a,$$

remember that, when $a > 0$,

$$x = \pm\sqrt{a}.$$

Do not silently discard one sign.

Red Flag: Inequalities

Whenever an inequality is:

multiplied

or

divided,

ask:

What is the sign of the multiplier?

Whenever it is:

squared,

ask:

What is the domain?

Red Flag: Functions

Whenever you see:

$$f(x) = f(y) \Rightarrow x = y,$$

ask:

Is f one-to-one on the relevant domain?

This immediately detects errors involving:

$$x^2,$$

$$|x|,$$

and:

$$\sin x.$$

Red Flag: “Therefore”

The most dangerous word in a proof can be:

therefore .

Every time you see it, mentally replace it with:

Why?

A proof is not validated by connective words.

It is validated by the mathematical relationship between the statements.

Challenge 1

A student writes:

$$x^2 = x.$$

Divide by x :

$$x = 1.$$

Find the missing solution.

Challenge 2

A student claims:

$$a^2 > b^2 \Rightarrow a > b.$$

Find a counterexample.

Challenge 3

A student claims:

$$\cos A = \cos B \Rightarrow A = B.$$

Find a counterexample.

Challenge 4

A student simplifies:

$$\frac{x^2 - 9}{x - 3}$$

to:

$$x + 3.$$

What condition must be retained?

Challenge 5

A student claims:

$$\sqrt{(x - 2)^2} = x - 2.$$

When is this correct?

Challenge 6

Suppose:

$$A \Rightarrow B$$

and B is false.

A student concludes:

A is false.

Is the deduction valid?

Challenge 7

Suppose:

$$A \Rightarrow B$$

and A is false.

A student concludes:

B is false.

Valid?

Challenge 7 (continued)

The antecedent is false but the consequent is true.

Challenge 8

A student tests a statement for:

$$n = 1, 2, \dots, 100$$

and finds that it is always true.

Have they proved it for every positive integer?

Correct Answer, Incorrect Proof

A statement may be true even when the proposed proof is wrong.
For example, a student might reach the correct conclusion using an invalid step.
Therefore distinguish:

truth of the conclusion

from:

validity of the argument.

A true conclusion does not repair a faulty proof.

False Conclusion, Useful Diagnosis

Likewise, when a proof reaches something absurd such as:

$$1 = 2,$$

do not merely say:

Obviously wrong.

Instead ask:

Which earlier step allowed the absurdity to enter?

The contradiction is evidence that something went wrong.
The mathematical task is locating it.

Conditions Are Part of Mathematics

Statements such as:

$$\frac{ab}{a} = b$$

are incomplete without considering:

$$a \neq 0.$$

Similarly:

$$\sqrt{x^2} = x$$

needs:

$$x \geq 0.$$

And:

$$x < y \Rightarrow x^2 < y^2$$

Conditions Are Part of Mathematics (continued)

needs appropriate restrictions.

Mathematical Maturity

Conditions are not technical decorations. They determine whether a mathematical statement is actually true.

Think Like a Skeptic

When reading a proof, temporarily assume that the writer may have made a mistake.

Try to break each important deduction.

Ask:

What happens at zero?

What happens for negative values?

What happens at the boundary?

Is there another branch or solution?

Can I construct a counterexample?

This is mathematical skepticism in a productive form.

Error Type 1: Invalid Implication

Do not assume:

$$A \Rightarrow B$$

gives:

$$B \Rightarrow A.$$

The converse requires independent justification.

But remember:

$$A \Rightarrow B \iff \neg B \Rightarrow \neg A.$$

The contrapositive is valid.

Error Type 2: Illegal Algebra

Watch for:

division by zero,

invalid cancellation,

lost solutions,

and

extraneous solutions.

Every algebraic transformation should be checked for reversibility and domain conditions.

Error Type 3: Domain and Sign

Check:

denominators,

square roots,

logarithms,

inequality multipliers,

and:

function domains.

Many apparently algebraic mistakes are really domain mistakes.

Error Type 4: Non-Unique Inverses

Do not automatically infer:

$$f(x) = f(y) \Rightarrow x = y.$$

Ask whether f is one-to-one.

In particular:

$$x^2 = y^2$$

does not imply $x = y$,
and:

$$\sin A = \sin B$$

does not imply $A = B$ in general.

Error Type 5: Evidence vs Proof

Examples can:

suggest

a conjecture.

Counterexamples can:

destroy

a universal conjecture.

But successful examples alone generally cannot:

prove

a universal statement.

The Complete Audit

When you see a purported proof:

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E,$$

do not read it merely as a story.

Test:

$$A \Rightarrow B?$$

Then:

$$B \Rightarrow C?$$

Then:

$$C \Rightarrow D?$$

Then:

The Complete Audit (continued)

$$D \Rightarrow E?$$

The first failed implication is where the proof breaks.

Self-Check

Before completing this masterclass, make sure you can:

- ❶ Locate the first invalid step in a proof.
- ❷ Detect invalid cancellation.
- ❸ Detect hidden division by zero.
- ❹ Distinguish a converse from a contrapositive.
- ❺ Recognise lost and extraneous solutions.
- ❻ Handle square and square-root ambiguities.
- ❼ Check signs in inequalities.
- ❽ Detect missing domain restrictions.
- ❾ Recognise non-injective function errors.
- ❿ Explain why $\sin A = \sin B$ need not imply $A = B$.
- ⓫ Use counterexamples to expose invalid deductions.
- ⓫ Repair an argument by identifying a missing condition.

What Paper 2 Has Built

Across this masterclass we have moved from:

Statements

to:

Logical Connections

to:

Quantifiers

to:

Proof

to:

Chains of Reasoning

What Paper 2 Has Built (continued)

and finally:

Proof Criticism.

The Progression

You should now be increasingly comfortable asking:

What exactly does this statement mean?

What follows from it?

What does not follow?

How can I prove it?

How can I disprove it?

and:

Where does this argument fail?

The Deeper Goal

Paper 2 is not fundamentally a collection of logical vocabulary.
The deeper objective is to become comfortable with:

precise mathematical reasoning.

That means being able to:

construct

an argument,

analyse

an argument,
and:

challenge

an argument.

Final Principle

A proof is not correct because it looks convincing.

It is correct because every required step is valid.

Check the Assumptions

Check the Conditions

Check Every Implication

Do Not Trust the Proof. Verify the Proof.

Mathematics Elevate Academy
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