

TMUA Mathematics Masterclass

Part 1: Mathematical Thinking, Indices, Surds and Quadratics

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About This Video Series

This series is designed to develop the mathematical knowledge, reasoning habits, and problem-solving fluency useful for the **Test of Mathematics for University Admission (TMUA)**.

Our Objective

The goal is not merely to revise school mathematics.

The goal is to learn how to

recognise structure → choose an efficient idea → reason accurately

under examination conditions.

This is an independent educational resource and is not affiliated with or endorsed by the organisations responsible for administering the TMUA.

What Will We Learn in Part 1?

- 1 What TMUA mathematics is really testing
- 2 Knowledge versus mathematical performance
- 3 How to think about unfamiliar problems
- 4 Algebra as a language of structure
- 5 Laws of indices
- 6 Fractional powers
- 7 Negative and zero powers
- 8 Rational exponents
- 9 Surds and exact values
- 10 Rationalising denominators
- 11 Quadratic structure
- 12 Completing the square
- 13 Roots, graphs and the discriminant

TMUA Is Not Simply a “Harder School Exam”

A common misconception is:

TMUA difficulty = more advanced mathematics.

That is not the most useful way to think about the examination.

TMUA Thinking

The challenge often comes from being asked to use familiar mathematics in an unfamiliar configuration.

You may know every formula appearing in a problem and still need considerable thought to determine:

- what information matters;
- what representation is useful;
- what can be deduced quickly;
- which approach avoids unnecessary calculation.

Mathematical Knowledge vs Mathematical Performance

There is an important distinction between

knowing mathematics

and

using mathematics effectively.

Suppose you know

$$a^m a^n = a^{m+n}.$$

That is **knowledge**.

But if a problem presents

$$2^{x+1} + 2^x = 48,$$

can you immediately recognize that

Mathematical Knowledge vs Mathematical Performance (continued)

$$2^{x+1} = 2 \cdot 2^x$$

and therefore write

$$3 \cdot 2^x = 48?$$

That is **mathematical performance**.

The Central TMUA Skill

Recognition before calculation

Strong problem solvers constantly ask:

- What structure is hidden here?
- Have I seen an equivalent mathematical form before?
- Can I transform the problem?
- Is there symmetry?
- Can I factor something?
- Can I use a graph?
- Can I eliminate impossible cases?
- Do I actually need the exact value?

TMUA Thinking

The fastest solution is frequently obtained by understanding the structure of the question before performing calculations.

The Three Layers of TMUA Preparation

Layer 1

Knowledge

Definitions
Formulae
Algebraic techniques
Standard graphs

Layer 2

Reasoning

Connections
Structure
Deduction
Representation

Layer 3

Performance

Speed
Accuracy
Selection
Decision-making

A complete preparation strategy must develop all three.

How Should You Use Past Papers?

Past papers should not initially be treated merely as collections of exercises. They are also a way of learning the **language of the exam**.

A productive cycle is

Attempt → Analyse → Repair → Repeat

After every difficult problem, ask:

- ① What knowledge was required?
- ② What was the key observation?
- ③ Why did I miss it?
- ④ Was my method unnecessarily long?
- ⑤ Could I recognise the idea faster next time?

Timed Practice Matters

Knowing how to solve a question eventually is different from solving it efficiently under examination conditions.

During timed practice, develop the ability to decide:

- Should I continue with this approach?
- Is there a shorter route?
- Am I doing algebra that the problem does not require?
- Can the answer choices help?
- Can I estimate or bound the answer?
- Should I temporarily leave this question?

Common Trap

Do not measure preparation only by the number of questions completed. Measure whether your **recognition, reasoning and efficiency** are improving.

Part I of the Mathematics: Algebra and Functions

Algebra is central to TMUA because it provides the language through which many other topics are expressed.

In this part of the course we will eventually study:

- indices;
- surds;
- quadratics;
- simultaneous equations;
- inequalities;
- polynomials;
- functions;
- transformations;
- algebraic manipulation.

Key Idea

Strong algebra reduces the amount of conscious effort required during harder problems.

Indices: More Than Rules to Memorise

Consider

$$a^5.$$

For a positive integer exponent,

$$a^5 = a \cdot a \cdot a \cdot a \cdot a.$$

The notation compresses repeated multiplication into a compact symbolic form.

This is important because good notation does more than save space.

It allows us to **see structure**.

For example,

$$a^3 a^4$$

immediately suggests combining repeated factors.

Deriving the Product Law

Start with

$$a^m a^n.$$

For positive integers m, n ,

$$a^m = \underbrace{a \cdot a \cdots a}_{m \text{ factors}},$$

and

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}.$$

Multiplying gives

$$\underbrace{a \cdot a \cdots a}_{m+n \text{ factors}}.$$

Therefore

Deriving the Product Law (continued)

$$\boxed{a^m a^n = a^{m+n}}.$$

TMUA Thinking

Do not think of this merely as “add the powers.”
Understand why the powers add.

The Product Law in Action

Simplify

$$x^7 x^{-3}.$$

Using

$$x^m x^n = x^{m+n},$$

we obtain

$$x^7 x^{-3} = x^{7+(-3)} = x^4.$$

Now consider

$$2^{x+1} 2^{3-x}.$$

Without knowing x ,

The Product Law in Action (continued)

$$2^{x+1} 2^{3-x} = 2^{(x+1)+(3-x)} = 2^4 = 16.$$

Key Idea

Indices can make complicated-looking expressions collapse immediately when their structure is recognised.

Fractional Powers: What Should $a^{1/n}$ Mean?

Suppose we want the product law

$$a^m a^n = a^{m+n}$$

to continue working for fractional exponents.
Consider

$$a^{1/3} a^{1/3} a^{1/3}.$$

The product law gives

$$a^{1/3+1/3+1/3} = a.$$

Therefore $a^{1/3}$ must be a number which, when cubed, produces a .
Hence

$$a^{1/3} = \sqrt[3]{a}.$$

More generally,

Fractional Powers: What Should $a^{1/n}$ Mean? (continued)

$$a^{1/n} = \sqrt[n]{a}.$$

General Rational Exponents

Combining roots and powers gives

$$a^{m/n}.$$

We can interpret this as

$$a^{m/n} = \left(a^{1/n}\right)^m.$$

Therefore

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

under the usual conditions being considered.

For example,

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9.$$

Negative Powers

What should

$$a^{-m}$$

mean?

We want the exponent laws to remain consistent.

Observe

$$a^m a^{-m} = a^{m-m} = a^0.$$

If $a^0 = 1$, then

$$a^m a^{-m} = 1.$$

Therefore

$$a^{-m} = \frac{1}{a^m}.$$

Negative Powers (continued)

For example,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

Why Is $a^0 = 1$?

Using the quotient law,

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But provided $a \neq 0$,

$$\frac{a^m}{a^m} = 1.$$

Hence

$$\boxed{a^0 = 1}.$$

Alternatively,

$$a^2 a^{-2} = a^0,$$

while

Why Is $a^0 = 1$? (continued)

$$a^2 a^{-2} = a^2 \frac{1}{a^2} = 1.$$

Again,

$$a^0 = 1.$$

The Core Index Laws

For appropriate values of the variables:

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^0 = 1$$

$$a^{1/n} = \sqrt[n]{a}$$

The Core Index Laws (continued)

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

A Very Important Notational Distinction

Compare

$$(a^m)^n$$

with

$$a^{m^n}.$$

They are not the same expression.
For example,

$$(a^3)^2 = a^6,$$

whereas

$$a^{3^2} = a^9.$$

A Very Important Notational Distinction (continued)

Common Trap

Parentheses matter.

Never manipulate powers by visual pattern alone. Read the mathematical structure carefully.

TMUA-Style Example 1

Suppose

$$2^x + 2^{x+1} = 48.$$

Find x .

The inefficient approach would be to try values of x .

Instead, recognise

$$2^{x+1} = 2 \cdot 2^x.$$

Therefore

$$2^x + 2(2^x) = 48,$$

so

$$3(2^x) = 48.$$

TMUA-Style Example 1 (continued)

Hence

$$2^x = 16 = 2^4,$$

and therefore

$$\boxed{x = 4}.$$

TMUA-Style Example 2

Simplify

$$\frac{9^{x+1}}{3^{2x-1}}.$$

Since

$$9 = 3^2,$$

we have

$$9^{x+1} = (3^2)^{x+1} = 3^{2x+2}.$$

Therefore

$$\frac{3^{2x+2}}{3^{2x-1}} = 3^{(2x+2)-(2x-1)} = 3^3.$$

Thus

TMUA-Style Example 2 (continued)

$$\boxed{27}.$$

TMUA Thinking

The variable disappeared.

That is a strong signal that structural simplification was the intended route.

TMUA-Style Example 3

Evaluate

$$16^{3/4}.$$

Use

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

Therefore

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3.$$

Since

$$\sqrt[4]{16} = 2,$$

we obtain

$$16^{3/4} = 2^3 = 8.$$

TMUA-Style Example 3 (continued)

Hence

$$\boxed{8}.$$

Challenge: Think Before Calculating

Suppose

$$x^{1/2} = 4.$$

Then clearly

$$x = 16.$$

But now suppose

$$x^{2/3} = 4.$$

How should we think?

Since

$$x^{2/3} = \left(x^{1/3}\right)^2,$$

we have

Challenge: Think Before Calculating (continued)

$$(\sqrt[3]{x})^2 = 4.$$

This gives

$$\sqrt[3]{x} = \pm 2.$$

Therefore

$$x = 8 \quad \text{or} \quad x = -8.$$

Common Trap

Reversing an even power may introduce both positive and negative possibilities.

What Is a Surd?

A surd is an exact expression involving a root that cannot be simplified to a rational number. Examples include

$$\sqrt{2}, \quad 3 + \sqrt{5}, \quad 2\sqrt{7}.$$

But

$$\sqrt{64} = 8$$

is not left as a surd because it simplifies exactly.

What Is a Surd? (continued)

Key Idea

Surds allow us to preserve exact information.
For example,

$$\sqrt{2}$$

is exact, whereas

$$1.414$$

is only an approximation.

Simplifying Surds

Consider

$$\sqrt{50}.$$

Factor 50 so that one factor is a perfect square:

$$50 = 25 \cdot 2.$$

Therefore

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}.$$

Thus

$$\boxed{\sqrt{50} = 5\sqrt{2}}.$$

The key skill is recognising useful square factors quickly.

Another Surd Example

Simplify

$$\sqrt{72}.$$

Since

$$72 = 36 \cdot 2,$$

we obtain

$$\sqrt{72} = \sqrt{36} \sqrt{2} = 6\sqrt{2}.$$

Alternatively,

$$72 = 9 \cdot 8,$$

so

Another Surd Example (continued)

$$\sqrt{72} = 3\sqrt{8} = 3(2\sqrt{2}) = 6\sqrt{2}.$$

TMUA Thinking

Different valid routes should lead to the same simplified form. Choose the factorisation you recognise fastest.

Multiplying Expressions Containing Surds

Consider

$$(2 + 3\sqrt{5})^2.$$

Use

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Hence

$$(2 + 3\sqrt{5})^2 = 4 + 12\sqrt{5} + 9(5).$$

Therefore

$$(2 + 3\sqrt{5})^2 = 49 + 12\sqrt{5}.$$

So

$$(2 + 3\sqrt{5})^2 = 49 + 12\sqrt{5}.$$

The Reverse Problem Is More Interesting

Suppose you encounter

$$49 + 12\sqrt{5}.$$

Can you recognise it as a perfect square?

Assume

$$49 + 12\sqrt{5} = (a + b\sqrt{5})^2.$$

Expanding,

$$(a + b\sqrt{5})^2 = a^2 + 5b^2 + 2ab\sqrt{5}.$$

Compare coefficients:

$$a^2 + 5b^2 = 49,$$

and

The Reverse Problem Is More Interesting (continued)

$$2ab = 12.$$

Thus

$$ab = 6.$$

Trying simple factor pairs gives

$$a = 2, \quad b = 3.$$

Hence

$$49 + 12\sqrt{5} = (2 + 3\sqrt{5})^2.$$

A Subtle Square-Root Trap

Consider

$$\sqrt{49 - 12\sqrt{5}}.$$

We notice

$$49 - 12\sqrt{5} = (2 - 3\sqrt{5})^2.$$

But

$$\sqrt{(2 - 3\sqrt{5})^2} \neq 2 - 3\sqrt{5}$$

because $2 - 3\sqrt{5} < 0$.

Recall

$$\sqrt{x^2} = |x|.$$

Therefore

A Subtle Square-Root Trap (continued)

$$\sqrt{49 - 12\sqrt{5}} = |2 - 3\sqrt{5}| = 3\sqrt{5} - 2.$$

Common Trap

The principal square root is non-negative.

Rationalising a Simple Denominator

Consider

$$\frac{1}{\sqrt{5}}.$$

Multiply numerator and denominator by $\sqrt{5}$:

$$\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

Hence

$$\boxed{\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}}.$$

More generally,

$$\boxed{\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}}$$

Rationalising a Simple Denominator (continued)

for suitable positive a .

Conjugates

Suppose the denominator is

$$a + b\sqrt{c}.$$

Its conjugate is

$$a - b\sqrt{c}.$$

Why is this useful?

Because

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c.$$

The surd disappears.

This comes from the difference-of-squares identity

$$(u + v)(u - v) = u^2 - v^2.$$

Rationalising a Binomial Denominator

Simplify

$$\frac{3}{2 + \sqrt{5}}.$$

Multiply by the conjugate:

$$\frac{3}{2 + \sqrt{5}} \cdot \frac{2 - \sqrt{5}}{2 - \sqrt{5}}.$$

Then

$$= \frac{3(2 - \sqrt{5})}{4 - 5}$$

and therefore

$$= -3(2 - \sqrt{5}).$$

Hence

Rationalising a Binomial Denominator (continued)

$$\boxed{\frac{3}{2 + \sqrt{5}} = 3\sqrt{5} - 6}.$$

Why Conjugates Matter Beyond Surds

The identity

$$(a + b)(a - b) = a^2 - b^2$$

appears throughout mathematics.

It can help with

- rationalising denominators;
- simplifying radicals;
- limits;
- algebraic factorisation;
- trigonometric manipulation;
- complex numbers later in mathematics.

TMUA Thinking

Strong candidates build a library of structural identities and learn to recognise when each identity is useful.

Why Quadratics Matter So Much

A quadratic function has the general form

$$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

Quadratics are particularly important because they connect:

- algebra;
- equations;
- inequalities;
- graphs;
- transformations;
- optimisation;
- calculus.

They provide an ideal setting for learning how different mathematical representations describe the same object.

Three Useful Forms of a Quadratic

A quadratic may appear in several forms.

Expanded form

$$ax^2 + bx + c.$$

Factorised form

$$a(x - r_1)(x - r_2).$$

Completed-square form

$$a(x - h)^2 + k.$$

Each form reveals different information.

Form	Information revealed quickly
Expanded	y-intercept, coefficients
Factorised	roots / x-intercepts
Completed square	vertex and symmetry

Completing the Square

Consider

$$x^2 + bx + c.$$

Start with

$$x^2 + bx.$$

Observe

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}.$$

Therefore

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}.$$

Hence

Completing the Square (continued)

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.$$

What Does Completing the Square Tell Us?

For

$$y = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4},$$

the squared term satisfies

$$\left(x + \frac{b}{2}\right)^2 \geq 0.$$

Therefore the minimum occurs when

$$x = -\frac{b}{2}.$$

The corresponding y-value is

$$c - \frac{b^2}{4}.$$

What Does Completing the Square Tell Us? (continued)

Thus the vertex is

$$\left(-\frac{b}{2}, c - \frac{b^2}{4}\right).$$

This connects algebra directly to geometry.

The General Quadratic

For

$$f(x) = ax^2 + bx + c,$$

completing the square gives

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

Therefore the axis of symmetry is

$$x = -\frac{b}{2a}.$$

The vertex occurs at

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right).$$

The General Quadratic (continued)

Notice the appearance of

$$b^2 - 4ac.$$

This quantity will become extremely important.

The Discriminant

Define

$$\Delta = b^2 - 4ac.$$

The quadratic formula is

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

The sign of Δ determines the number of real roots.

$\Delta > 0$	$\Rightarrow$	two distinct real roots,
$\Delta = 0$	$\Rightarrow$	one repeated real root,
$\Delta < 0$	$\Rightarrow$	no real roots.

The Discriminant Has a Geometric Meaning

Roots satisfy

$$f(x) = 0.$$

Graphically, these are exactly the points where

$$y = f(x)$$

meets the x -axis.

Therefore:

$$\Delta > 0$$

means the parabola intersects the x -axis twice.

$$\Delta = 0$$

means the parabola touches the axis once.

The Discriminant Has a Geometric Meaning (continued)

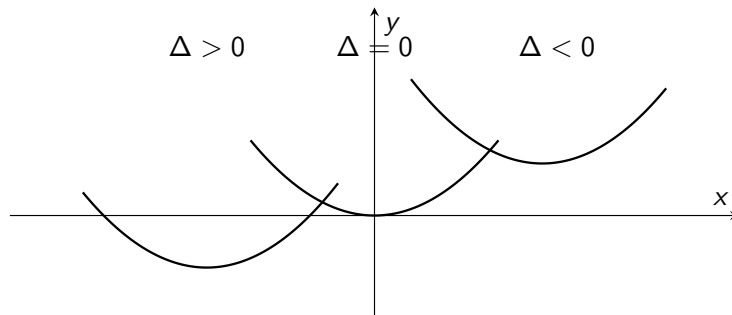
$$\Delta < 0$$

means the parabola does not meet the x -axis.

Key Idea

The discriminant is not merely part of a formula.
It describes the geometry of the quadratic.

Visualising the Three Cases



TMUA-Style Quadratic Reasoning

For what values of k does

$$x^2 - 4x + k = 0$$

have two distinct real roots?

We do not need to solve the quadratic.

Its discriminant is

$$\Delta = (-4)^2 - 4(1)(k) = 16 - 4k.$$

For two distinct real roots,

$$\Delta > 0.$$

Thus

$$16 - 4k > 0,$$

so

TMUA-Style Quadratic Reasoning (continued)

$$k < 4.$$

TMUA Thinking

Answer the question that was actually asked.
Finding the roots would have been unnecessary work.

Another Structural Question

Suppose

$$f(x) = x^2 - 6x + 13.$$

Without solving $f(x) = 0$, determine whether $f(x)$ can ever be negative.
Complete the square:

$$f(x) = x^2 - 6x + 9 + 4 = (x - 3)^2 + 4.$$

Since

$$(x - 3)^2 \geq 0,$$

we have

$$f(x) \geq 4.$$

Therefore

Another Structural Question (continued)

$f(x)$ is never negative.

The minimum value is 4.

One Expression, Several Viewpoints

Consider

$$x^2 - 6x + 5.$$

Expanded form

$$x^2 - 6x + 5.$$

Factorised form

$$(x - 1)(x - 5).$$

Therefore the roots are

$$1, 5.$$

Completed-square form

$$(x - 3)^2 - 4.$$

One Expression, Several Viewpoints (continued)

Therefore the vertex is

$$(3, -4).$$

All three expressions describe the **same quadratic**.

What Should You Take Away From Part 1?

- ❶ TMUA rewards mathematical reasoning, not merely formula recall.
- ❷ Familiar mathematics may appear in unfamiliar forms.
- ❸ Index laws should be understood structurally.
- ❹ Fractional powers connect indices with roots.
- ❺ Negative powers represent reciprocals.
- ❻ Surds preserve exact values.
- ❼ Conjugates exploit difference of squares.
- ❽ Quadratic forms reveal different information.
- ❾ Completing the square connects algebra with geometry.
- ❿ The discriminant describes the root structure of a quadratic.

The Bigger Lesson

Algebra $\longleftrightarrow$ Geometry $\longleftrightarrow$ Reasoning

TMUA preparation becomes much more powerful when topics stop being isolated chapters. For example,

$$ax^2 + bx + c$$

is simultaneously

- an algebraic expression,
- an equation when set equal to zero,
- a function,
- a parabola,
- an object with roots,
- an object with a maximum or minimum.

Before Moving to Part 2

Make sure you can answer these questions confidently:

Self-Check

- 1 Why does $a^m a^n = a^{m+n}$?
- 2 Why is $a^{-m} = 1/a^m$?
- 3 What does $a^{m/n}$ mean?
- 4 Why is $\sqrt{x^2} = |x|$?
- 5 Why does multiplying conjugates remove a square-root term?
- 6 What information does each form of a quadratic reveal?
- 7 Why does $b^2 - 4ac$ determine the number of real roots?
- 8 How is the discriminant visible on the graph?

TMUA Mathematics Masterclass

Part 2: Algebra and Functions — Advanced Structure

We will continue with:

- simultaneous equations;
- linear–quadratic intersections;
- inequalities;
- polynomial structure;
- algebraic division;
- factor and remainder ideas;
- functions;
- domain and range;
- composite functions;
- inverse functions;
- transformations and graph reasoning.

Think Structurally.

Do not ask only

“Which formula should I use?”

Ask

“What is the mathematics telling me?”

Mathematics Elevate Academy