

TMUA Mathematics Masterclass

Part 1: Mathematical Thinking, Indices, Surds and Quadratics

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About This Video Series

This series is designed to develop the mathematical knowledge, reasoning habits, and problem-solving fluency useful for the **Test of Mathematics for University Admission (TMUA)**.

Our Objective

The goal is not merely to revise school mathematics.

The goal is to learn how to

recognise structure → choose an efficient idea → reason accurately

under examination conditions.

This is an independent educational resource and is not affiliated with or endorsed by the organisations responsible for administering the TMUA.

What Will We Learn in Part 1?

- 1 What TMUA mathematics is really testing
- 2 Knowledge versus mathematical performance
- 3 How to think about unfamiliar problems
- 4 Algebra as a language of structure
- 5 Laws of indices
- 6 Fractional powers
- 7 Negative and zero powers
- 8 Rational exponents
- 9 Surds and exact values
- 10 Rationalising denominators
- 11 Quadratic structure
- 12 Completing the square
- 13 Roots, graphs and the discriminant

TMUA Is Not Simply a “Harder School Exam”

A common misconception is:

TMUA difficulty = more advanced mathematics.

That is not the most useful way to think about the examination.

TMUA Thinking

The challenge often comes from being asked to use familiar mathematics in an unfamiliar configuration.

You may know every formula appearing in a problem and still need considerable thought to determine:

- what information matters;
- what representation is useful;
- what can be deduced quickly;
- which approach avoids unnecessary calculation.

Mathematical Knowledge vs Mathematical Performance

There is an important distinction between

knowing mathematics

and

using mathematics effectively.

Suppose you know

$$a^m a^n = a^{m+n}.$$

That is **knowledge**.

But if a problem presents

$$2^{x+1} + 2^x = 48,$$

can you immediately recognize that

Mathematical Knowledge vs Mathematical Performance (continued)

$$2^{x+1} = 2 \cdot 2^x$$

and therefore write

$$3 \cdot 2^x = 48?$$

That is **mathematical performance**.

The Central TMUA Skill

Recognition before calculation

Strong problem solvers constantly ask:

- What structure is hidden here?
- Have I seen an equivalent mathematical form before?
- Can I transform the problem?
- Is there symmetry?
- Can I factor something?
- Can I use a graph?
- Can I eliminate impossible cases?
- Do I actually need the exact value?

TMUA Thinking

The fastest solution is frequently obtained by understanding the structure of the question before performing calculations.

The Three Layers of TMUA Preparation

Layer 1

Knowledge

Definitions
Formulae
Algebraic techniques
Standard graphs

Layer 2

Reasoning

Connections
Structure
Deduction
Representation

Layer 3

Performance

Speed
Accuracy
Selection
Decision-making

A complete preparation strategy must develop all three.

How Should You Use Past Papers?

Past papers should not initially be treated merely as collections of exercises. They are also a way of learning the **language of the exam**.

A productive cycle is

Attempt → Analyse → Repair → Repeat

After every difficult problem, ask:

- ① What knowledge was required?
- ② What was the key observation?
- ③ Why did I miss it?
- ④ Was my method unnecessarily long?
- ⑤ Could I recognise the idea faster next time?

Timed Practice Matters

Knowing how to solve a question eventually is different from solving it efficiently under examination conditions.

During timed practice, develop the ability to decide:

- Should I continue with this approach?
- Is there a shorter route?
- Am I doing algebra that the problem does not require?
- Can the answer choices help?
- Can I estimate or bound the answer?
- Should I temporarily leave this question?

Common Trap

Do not measure preparation only by the number of questions completed. Measure whether your **recognition, reasoning and efficiency** are improving.

Part I of the Mathematics: Algebra and Functions

Algebra is central to TMUA because it provides the language through which many other topics are expressed.

In this part of the course we will eventually study:

- indices;
- surds;
- quadratics;
- simultaneous equations;
- inequalities;
- polynomials;
- functions;
- transformations;
- algebraic manipulation.

Key Idea

Strong algebra reduces the amount of conscious effort required during harder problems.

Indices: More Than Rules to Memorise

Consider

$$a^5.$$

For a positive integer exponent,

$$a^5 = a \cdot a \cdot a \cdot a \cdot a.$$

The notation compresses repeated multiplication into a compact symbolic form.

This is important because good notation does more than save space.

It allows us to **see structure**.

For example,

$$a^3 a^4$$

immediately suggests combining repeated factors.

Deriving the Product Law

Start with

$$a^m a^n.$$

For positive integers m, n ,

$$a^m = \underbrace{a \cdot a \cdots a}_{m \text{ factors}},$$

and

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}.$$

Multiplying gives

$$\underbrace{a \cdot a \cdots a}_{m+n \text{ factors}}.$$

Therefore

Deriving the Product Law (continued)

$$\boxed{a^m a^n = a^{m+n}}.$$

TMUA Thinking

Do not think of this merely as “add the powers.”
Understand why the powers add.

The Product Law in Action

Simplify

$$x^7 x^{-3}.$$

Using

$$x^m x^n = x^{m+n},$$

we obtain

$$x^7 x^{-3} = x^{7+(-3)} = x^4.$$

Now consider

$$2^{x+1} 2^{3-x}.$$

Without knowing x ,

The Product Law in Action (continued)

$$2^{x+1} 2^{3-x} = 2^{(x+1)+(3-x)} = 2^4 = 16.$$

Key Idea

Indices can make complicated-looking expressions collapse immediately when their structure is recognised.

Fractional Powers: What Should $a^{1/n}$ Mean?

Suppose we want the product law

$$a^m a^n = a^{m+n}$$

to continue working for fractional exponents.
Consider

$$a^{1/3} a^{1/3} a^{1/3}.$$

The product law gives

$$a^{1/3+1/3+1/3} = a.$$

Therefore $a^{1/3}$ must be a number which, when cubed, produces a .
Hence

$$a^{1/3} = \sqrt[3]{a}.$$

More generally,

Fractional Powers: What Should $a^{1/n}$ Mean? (continued)

$$a^{1/n} = \sqrt[n]{a}.$$

General Rational Exponents

Combining roots and powers gives

$$a^{m/n}.$$

We can interpret this as

$$a^{m/n} = \left(a^{1/n}\right)^m.$$

Therefore

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

under the usual conditions being considered.

For example,

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9.$$

Negative Powers

What should

$$a^{-m}$$

mean?

We want the exponent laws to remain consistent.

Observe

$$a^m a^{-m} = a^{m-m} = a^0.$$

If $a^0 = 1$, then

$$a^m a^{-m} = 1.$$

Therefore

$$a^{-m} = \frac{1}{a^m}.$$

Negative Powers (continued)

For example,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

Why Is $a^0 = 1$?

Using the quotient law,

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But provided $a \neq 0$,

$$\frac{a^m}{a^m} = 1.$$

Hence

$$\boxed{a^0 = 1}.$$

Alternatively,

$$a^2 a^{-2} = a^0,$$

while

Why Is $a^0 = 1$? (continued)

$$a^2 a^{-2} = a^2 \frac{1}{a^2} = 1.$$

Again,

$$a^0 = 1.$$

The Core Index Laws

For appropriate values of the variables:

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^0 = 1$$

$$a^{1/n} = \sqrt[n]{a}$$

The Core Index Laws (continued)

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

A Very Important Notational Distinction

Compare

$$(a^m)^n$$

with

$$a^{m^n}.$$

They are not the same expression.
For example,

$$(a^3)^2 = a^6,$$

whereas

$$a^{3^2} = a^9.$$

A Very Important Notational Distinction (continued)

Common Trap

Parentheses matter.

Never manipulate powers by visual pattern alone. Read the mathematical structure carefully.

TMUA-Style Example 1

Suppose

$$2^x + 2^{x+1} = 48.$$

Find x .

The inefficient approach would be to try values of x .

Instead, recognise

$$2^{x+1} = 2 \cdot 2^x.$$

Therefore

$$2^x + 2(2^x) = 48,$$

so

$$3(2^x) = 48.$$

TMUA-Style Example 1 (continued)

Hence

$$2^x = 16 = 2^4,$$

and therefore

$$\boxed{x = 4}.$$

TMUA-Style Example 2

Simplify

$$\frac{9^{x+1}}{3^{2x-1}}.$$

Since

$$9 = 3^2,$$

we have

$$9^{x+1} = (3^2)^{x+1} = 3^{2x+2}.$$

Therefore

$$\frac{3^{2x+2}}{3^{2x-1}} = 3^{(2x+2)-(2x-1)} = 3^3.$$

Thus

TMUA-Style Example 2 (continued)

$$\boxed{27}.$$

TMUA Thinking

The variable disappeared.

That is a strong signal that structural simplification was the intended route.

TMUA-Style Example 3

Evaluate

$$16^{3/4}.$$

Use

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

Therefore

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3.$$

Since

$$\sqrt[4]{16} = 2,$$

we obtain

$$16^{3/4} = 2^3 = 8.$$

TMUA-Style Example 3 (continued)

Hence

$$\boxed{8}.$$

Challenge: Think Before Calculating

Suppose

$$x^{1/2} = 4.$$

Then clearly

$$x = 16.$$

But now suppose

$$x^{2/3} = 4.$$

How should we think?

Since

$$x^{2/3} = \left(x^{1/3}\right)^2,$$

we have

Challenge: Think Before Calculating (continued)

$$(\sqrt[3]{x})^2 = 4.$$

This gives

$$\sqrt[3]{x} = \pm 2.$$

Therefore

$$x = 8 \quad \text{or} \quad x = -8.$$

Common Trap

Reversing an even power may introduce both positive and negative possibilities.

What Is a Surd?

A surd is an exact expression involving a root that cannot be simplified to a rational number. Examples include

$$\sqrt{2}, \quad 3 + \sqrt{5}, \quad 2\sqrt{7}.$$

But

$$\sqrt{64} = 8$$

is not left as a surd because it simplifies exactly.

What Is a Surd? (continued)

Key Idea

Surds allow us to preserve exact information.
For example,

$$\sqrt{2}$$

is exact, whereas

$$1.414$$

is only an approximation.

Simplifying Surds

Consider

$$\sqrt{50}.$$

Factor 50 so that one factor is a perfect square:

$$50 = 25 \cdot 2.$$

Therefore

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}.$$

Thus

$$\boxed{\sqrt{50} = 5\sqrt{2}}.$$

The key skill is recognising useful square factors quickly.

Another Surd Example

Simplify

$$\sqrt{72}.$$

Since

$$72 = 36 \cdot 2,$$

we obtain

$$\sqrt{72} = \sqrt{36} \sqrt{2} = 6\sqrt{2}.$$

Alternatively,

$$72 = 9 \cdot 8,$$

so

Another Surd Example (continued)

$$\sqrt{72} = 3\sqrt{8} = 3(2\sqrt{2}) = 6\sqrt{2}.$$

TMUA Thinking

Different valid routes should lead to the same simplified form. Choose the factorisation you recognise fastest.

Multiplying Expressions Containing Surds

Consider

$$(2 + 3\sqrt{5})^2.$$

Use

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Hence

$$(2 + 3\sqrt{5})^2 = 4 + 12\sqrt{5} + 9(5).$$

Therefore

$$(2 + 3\sqrt{5})^2 = 49 + 12\sqrt{5}.$$

So

$$(2 + 3\sqrt{5})^2 = 49 + 12\sqrt{5}.$$

The Reverse Problem Is More Interesting

Suppose you encounter

$$49 + 12\sqrt{5}.$$

Can you recognise it as a perfect square?

Assume

$$49 + 12\sqrt{5} = (a + b\sqrt{5})^2.$$

Expanding,

$$(a + b\sqrt{5})^2 = a^2 + 5b^2 + 2ab\sqrt{5}.$$

Compare coefficients:

$$a^2 + 5b^2 = 49,$$

and

The Reverse Problem Is More Interesting (continued)

$$2ab = 12.$$

Thus

$$ab = 6.$$

Trying simple factor pairs gives

$$a = 2, \quad b = 3.$$

Hence

$$49 + 12\sqrt{5} = (2 + 3\sqrt{5})^2.$$

A Subtle Square-Root Trap

Consider

$$\sqrt{49 - 12\sqrt{5}}.$$

We notice

$$49 - 12\sqrt{5} = (2 - 3\sqrt{5})^2.$$

But

$$\sqrt{(2 - 3\sqrt{5})^2} \neq 2 - 3\sqrt{5}$$

because $2 - 3\sqrt{5} < 0$.

Recall

$$\sqrt{x^2} = |x|.$$

Therefore

A Subtle Square-Root Trap (continued)

$$\sqrt{49 - 12\sqrt{5}} = |2 - 3\sqrt{5}| = 3\sqrt{5} - 2.$$

Common Trap

The principal square root is non-negative.

Rationalising a Simple Denominator

Consider

$$\frac{1}{\sqrt{5}}.$$

Multiply numerator and denominator by $\sqrt{5}$:

$$\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

Hence

$$\boxed{\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}}.$$

More generally,

$$\boxed{\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}}$$

Rationalising a Simple Denominator (continued)

for suitable positive a .

Conjugates

Suppose the denominator is

$$a + b\sqrt{c}.$$

Its conjugate is

$$a - b\sqrt{c}.$$

Why is this useful?

Because

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c.$$

The surd disappears.

This comes from the difference-of-squares identity

$$(u + v)(u - v) = u^2 - v^2.$$

Rationalising a Binomial Denominator

Simplify

$$\frac{3}{2 + \sqrt{5}}.$$

Multiply by the conjugate:

$$\frac{3}{2 + \sqrt{5}} \cdot \frac{2 - \sqrt{5}}{2 - \sqrt{5}}.$$

Then

$$= \frac{3(2 - \sqrt{5})}{4 - 5}$$

and therefore

$$= -3(2 - \sqrt{5}).$$

Hence

Rationalising a Binomial Denominator (continued)

$$\boxed{\frac{3}{2 + \sqrt{5}} = 3\sqrt{5} - 6}.$$

Why Conjugates Matter Beyond Surds

The identity

$$(a + b)(a - b) = a^2 - b^2$$

appears throughout mathematics.

It can help with

- rationalising denominators;
- simplifying radicals;
- limits;
- algebraic factorisation;
- trigonometric manipulation;
- complex numbers later in mathematics.

TMUA Thinking

Strong candidates build a library of structural identities and learn to recognise when each identity is useful.

Why Quadratics Matter So Much

A quadratic function has the general form

$$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

Quadratics are particularly important because they connect:

- algebra;
- equations;
- inequalities;
- graphs;
- transformations;
- optimisation;
- calculus.

They provide an ideal setting for learning how different mathematical representations describe the same object.

Three Useful Forms of a Quadratic

A quadratic may appear in several forms.

Expanded form

$$ax^2 + bx + c.$$

Factorised form

$$a(x - r_1)(x - r_2).$$

Completed-square form

$$a(x - h)^2 + k.$$

Each form reveals different information.

Form	Information revealed quickly
Expanded	y-intercept, coefficients
Factorised	roots / x-intercepts
Completed square	vertex and symmetry

Completing the Square

Consider

$$x^2 + bx + c.$$

Start with

$$x^2 + bx.$$

Observe

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}.$$

Therefore

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}.$$

Hence

Completing the Square (continued)

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.$$

What Does Completing the Square Tell Us?

For

$$y = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4},$$

the squared term satisfies

$$\left(x + \frac{b}{2}\right)^2 \geq 0.$$

Therefore the minimum occurs when

$$x = -\frac{b}{2}.$$

The corresponding y-value is

$$c - \frac{b^2}{4}.$$

What Does Completing the Square Tell Us? (continued)

Thus the vertex is

$$\left(-\frac{b}{2}, c - \frac{b^2}{4}\right).$$

This connects algebra directly to geometry.

The General Quadratic

For

$$f(x) = ax^2 + bx + c,$$

completing the square gives

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

Therefore the axis of symmetry is

$$x = -\frac{b}{2a}.$$

The vertex occurs at

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right).$$

The General Quadratic (continued)

Notice the appearance of

$$b^2 - 4ac.$$

This quantity will become extremely important.

The Discriminant

Define

$$\Delta = b^2 - 4ac.$$

The quadratic formula is

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

The sign of Δ determines the number of real roots.

$\Delta > 0 \Rightarrow$	two distinct real roots,
$\Delta = 0 \Rightarrow$	one repeated real root,
$\Delta < 0 \Rightarrow$	no real roots.

The Discriminant Has a Geometric Meaning

Roots satisfy

$$f(x) = 0.$$

Graphically, these are exactly the points where

$$y = f(x)$$

meets the x -axis.

Therefore:

$$\Delta > 0$$

means the parabola intersects the x -axis twice.

$$\Delta = 0$$

means the parabola touches the axis once.

The Discriminant Has a Geometric Meaning (continued)

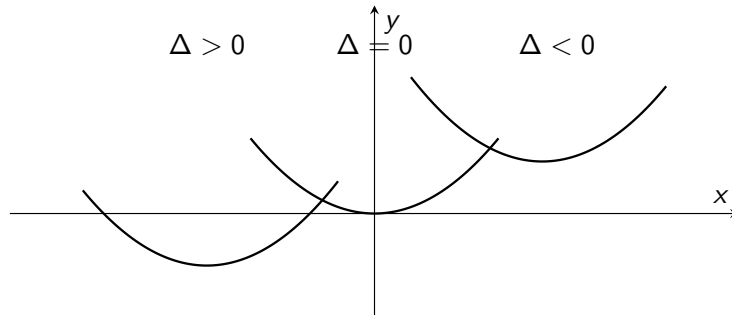
$$\Delta < 0$$

means the parabola does not meet the x -axis.

Key Idea

The discriminant is not merely part of a formula.
It describes the geometry of the quadratic.

Visualising the Three Cases



TMUA-Style Quadratic Reasoning

For what values of k does

$$x^2 - 4x + k = 0$$

have two distinct real roots?

We do not need to solve the quadratic.

Its discriminant is

$$\Delta = (-4)^2 - 4(1)(k) = 16 - 4k.$$

For two distinct real roots,

$$\Delta > 0.$$

Thus

$$16 - 4k > 0,$$

so

TMUA-Style Quadratic Reasoning (continued)

$$k < 4.$$

TMUA Thinking

Answer the question that was actually asked.
Finding the roots would have been unnecessary work.

Another Structural Question

Suppose

$$f(x) = x^2 - 6x + 13.$$

Without solving $f(x) = 0$, determine whether $f(x)$ can ever be negative.
Complete the square:

$$f(x) = x^2 - 6x + 9 + 4 = (x - 3)^2 + 4.$$

Since

$$(x - 3)^2 \geq 0,$$

we have

$$f(x) \geq 4.$$

Therefore

Another Structural Question (continued)

$f(x)$ is never negative.

The minimum value is 4.

One Expression, Several Viewpoints

Consider

$$x^2 - 6x + 5.$$

Expanded form

$$x^2 - 6x + 5.$$

Factorised form

$$(x - 1)(x - 5).$$

Therefore the roots are

$$1, 5.$$

Completed-square form

$$(x - 3)^2 - 4.$$

One Expression, Several Viewpoints (continued)

Therefore the vertex is

$$(3, -4).$$

All three expressions describe the **same quadratic**.

What Should You Take Away From Part 1?

- ❶ TMUA rewards mathematical reasoning, not merely formula recall.
- ❷ Familiar mathematics may appear in unfamiliar forms.
- ❸ Index laws should be understood structurally.
- ❹ Fractional powers connect indices with roots.
- ❺ Negative powers represent reciprocals.
- ❻ Surds preserve exact values.
- ❼ Conjugates exploit difference of squares.
- ❽ Quadratic forms reveal different information.
- ❾ Completing the square connects algebra with geometry.
- ❿ The discriminant describes the root structure of a quadratic.

The Bigger Lesson

Algebra $\longleftrightarrow$ Geometry $\longleftrightarrow$ Reasoning

TMUA preparation becomes much more powerful when topics stop being isolated chapters. For example,

$$ax^2 + bx + c$$

is simultaneously

- an algebraic expression,
- an equation when set equal to zero,
- a function,
- a parabola,
- an object with roots,
- an object with a maximum or minimum.

Before Moving to Part 2

Make sure you can answer these questions confidently:

Self-Check

- 1 Why does $a^m a^n = a^{m+n}$?
- 2 Why is $a^{-m} = 1/a^m$?
- 3 What does $a^{m/n}$ mean?
- 4 Why is $\sqrt{x^2} = |x|$?
- 5 Why does multiplying conjugates remove a square-root term?
- 6 What information does each form of a quadratic reveal?
- 7 Why does $b^2 - 4ac$ determine the number of real roots?
- 8 How is the discriminant visible on the graph?

TMUA Mathematics Masterclass

Part 2: Algebra and Functions — Advanced Structure

We will continue with:

- simultaneous equations;
- linear–quadratic intersections;
- inequalities;
- polynomial structure;
- algebraic division;
- factor and remainder ideas;
- functions;
- domain and range;
- composite functions;
- inverse functions;
- transformations and graph reasoning.

Think Structurally.

Do not ask only

“Which formula should I use?”

Ask

“What is the mathematics telling me?”

Mathematics Elevate Academy

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Part 2: Algebra and Functions — Structure, Equations and Graphs

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Where We Are

In Part 1 we developed several fundamental ideas:

- laws of indices;
- rational exponents;
- surds;
- exact manipulation;
- quadratic expressions;
- completing the square;
- roots and the discriminant.

Now we move from individual algebraic techniques to a broader idea:

How can algebra describe mathematical structure?

TMUA Thinking

TMUA questions frequently combine several elementary ideas inside one unfamiliar-looking problem.

Part 2 Roadmap

We will study:

- 1 Simultaneous equations
- 2 Linear–quadratic systems
- 3 Equations as intersections
- 4 Linear inequalities
- 5 Quadratic inequalities
- 6 Polynomial structure
- 7 Factorisation
- 8 Factor and remainder ideas
- 9 Functions and notation
- 10 Domain and range
- 11 Composite functions
- 12 Inverse functions
- 13 Graph transformations
- 14 Algebraic and graphical reasoning

What Does It Mean to Solve an Equation?

Consider

$$2x + 5 = 17.$$

We often say:

$$2x = 12$$

and therefore

$$x = 6.$$

But conceptually, solving an equation means finding all values of the variable for which the mathematical statement is true.

Key Idea

An equation is a condition.

A solution is a value satisfying that condition.

What Does It Mean to Solve an Equation? (continued)

This viewpoint becomes increasingly important when equations become nonlinear or involve several variables.

Equivalent Equations

Suppose

$$3x + 7 = 19.$$

Subtracting 7 from both sides gives

$$3x = 12.$$

Dividing both sides by 3 gives

$$x = 4.$$

Each step preserves the solution set.

We are replacing one equation by an **equivalent equation**.

$$3x + 7 = 19 \iff 3x = 12 \iff x = 4.$$

The symbol

Equivalent Equations (continued)



means “if and only if.”

A Warning About Algebraic Operations

Not every operation automatically preserves equivalence.
For example,

$$x = 2$$

certainly implies

$$x^2 = 4.$$

But

$$x^2 = 4$$

does not imply only

$$x = 2.$$

It also allows

A Warning About Algebraic Operations (continued)

$$x = -2.$$

Thus

$$x = 2 \implies x^2 = 4,$$

but not

$$x = 2 \iff x^2 = 4.$$

Common Trap

Some algebraic operations can introduce additional candidate solutions. Always check whether your transformations are reversible.

What Are Simultaneous Equations?

Consider

$$\begin{cases} x + y = 7, \\ x - y = 1. \end{cases}$$

We seek values of x and y satisfying **both equations simultaneously**.
Adding the equations gives

$$2x = 8,$$

so

$$x = 4.$$

Then

$$y = 3.$$

Therefore

What Are Simultaneous Equations? (continued)

$$(x, y) = (4, 3).$$

The Geometric Interpretation

The equations

$$x + y = 7$$

and

$$x - y = 1$$

describe two straight lines.

Their simultaneous solution is the point where the two lines intersect.

$$\text{Solving simultaneously} \iff \text{finding an intersection}$$

This idea generalises.

If

$$y = f(x)$$

The Geometric Interpretation (continued)

and

$$y = g(x),$$

then their intersections satisfy

$$\boxed{f(x) = g(x)}.$$

Linear–Quadratic Systems

Consider

$$y = x + 1$$

and

$$y = x^2 - 3x + 5.$$

At an intersection,

$$x + 1 = x^2 - 3x + 5.$$

Therefore

$$x^2 - 4x + 4 = 0.$$

Hence

$$(x - 2)^2 = 0,$$

Linear–Quadratic Systems (continued)

so

$$x = 2.$$

Then

$$y = 3.$$

Thus the graphs meet at

$$\boxed{(2, 3)}.$$

What Does a Repeated Root Mean Here?

We obtained

$$(x - 2)^2 = 0.$$

There is only one intersection.
Geometrically, the line

$$y = x + 1$$

is tangent to the parabola

$$y = x^2 - 3x + 5.$$

This gives another interpretation of the discriminant.
If solving

$$f(x) = g(x)$$

What Does a Repeated Root Mean Here? (continued)

produces a quadratic, then:

$$\Delta > 0$$

means two intersections,

$$\Delta = 0$$

means one touching point,
and

$$\Delta < 0$$

means no real intersections.

Parameter Problem

For what values of k does the line

$$y = k$$

intersect the parabola

$$y = x^2 - 4x + 7$$

at two distinct points?
Set them equal:

$$x^2 - 4x + 7 = k.$$

Hence

$$x^2 - 4x + (7 - k) = 0.$$

For two distinct real roots,

Parameter Problem (continued)

$$\Delta > 0.$$

Therefore

$$16 - 4(7 - k) > 0.$$

So

$$16 - 28 + 4k > 0,$$

giving

$$4k > 12.$$

Hence

$$\boxed{k > 3}.$$

Could We Have Seen This Without the Discriminant?

Complete the square:

$$x^2 - 4x + 7 = (x - 2)^2 + 3.$$

Therefore the parabola has minimum value

$$3.$$

A horizontal line $y = k$:

- misses the parabola if $k < 3$;
- touches it if $k = 3$;
- intersects twice if $k > 3$.

Could We Have Seen This Without the Discriminant? (continued)

Thus again,

$$k > 3.$$

TMUA Thinking

A strong problem solver looks for multiple representations.
The shortest route depends on how the problem is presented.

Equations and Inequalities Are Different

An equation such as

$$2x + 3 = 9$$

typically asks for particular values.

An inequality such as

$$2x + 3 < 9$$

usually describes an entire set of values.

Here,

$$2x < 6$$

and therefore

$$x < 3.$$

The answer is an interval rather than a single number.

The Sign-Reversal Rule

Consider

$$-2x < 6.$$

Dividing by -2 gives

$$x > -3.$$

The inequality reverses.

Why?

Because multiplication by a negative number reverses order.

For example,

$$2 < 5,$$

but multiplying by -1 gives

$$-2 > -5.$$

The Sign-Reversal Rule (continued)

Thus

$$a < b \implies -ca > -cb$$

for $c > 0$.

A Dangerous Inequality Move

Suppose

$$ax < a.$$

Can we simply divide by a ?

Not unless we know the sign of a .

If

$$a > 0,$$

then

$$x < 1.$$

But if

$$a < 0,$$

then

A Dangerous Inequality Move (continued)

$$x > 1.$$

And if

$$a = 0,$$

the original statement becomes

$$0 < 0,$$

which is false.

Common Trap

Never divide an inequality by an expression whose sign is unknown without considering cases.

Solving a Quadratic Inequality

Solve

$$x^2 - 5x + 6 > 0.$$

Factor:

$$(x - 2)(x - 3) > 0.$$

The critical values are

$$x = 2, \quad x = 3.$$

These divide the number line into

$$(-\infty, 2), \quad (2, 3), \quad (3, \infty).$$

We now determine the sign of the product in each interval.

Sign Analysis

Consider

$$(x - 2)(x - 3).$$

For $x < 2$:

$$x - 2 < 0, \quad x - 3 < 0,$$

so the product is positive.

For $2 < x < 3$:

$$x - 2 > 0, \quad x - 3 < 0,$$

so the product is negative.

For $x > 3$:

$$x - 2 > 0, \quad x - 3 > 0,$$

so the product is positive.

Sign Analysis (continued)

Therefore

$$x < 2 \quad \text{or} \quad x > 3.$$

The Graphical Interpretation

The inequality

$$x^2 - 5x + 6 > 0$$

asks:

When is $y = x^2 - 5x + 6$ above the x-axis?

Since

$$y = (x - 2)(x - 3),$$

the graph crosses the axis at

$$x = 2, \quad x = 3.$$

Because the coefficient of x^2 is positive, the parabola opens upward.
Therefore it lies above the axis when

The Graphical Interpretation (continued)

$$x < 2 \quad \text{or} \quad x > 3.$$

A More Efficient Inequality

Solve

$$x^2 - 6x + 10 > 0.$$

Instead of searching for roots, complete the square:

$$x^2 - 6x + 10 = (x - 3)^2 + 1.$$

Since

$$(x - 3)^2 \geq 0,$$

we have

$$(x - 3)^2 + 1 \geq 1 > 0.$$

Therefore the inequality is true for

$$\text{all real } x.$$

A More Efficient Inequality (continued)

TMUA Thinking

Before solving an inequality mechanically, inspect its structure.

Polynomial Structure

A polynomial in x has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If

$$a_n \neq 0,$$

then the degree is n .

Examples:

$$3x^4 - 2x + 7$$

has degree 4,
while

$$x^2 - 5x + 6$$

has degree 2.

Polynomial Structure (continued)

The degree gives important information about the possible algebraic and graphical behaviour of the polynomial.

Roots and Factors

Suppose

$$P(a) = 0.$$

Then a is a root of

$$P(x) = 0.$$

The corresponding linear expression

$$x - a$$

is a factor of $P(x)$.

Thus

$$P(a) = 0 \iff (x - a) \text{ is a factor of } P(x)$$

This is one of the most useful connections in polynomial algebra.

Example: Using a Known Root

Suppose

$$P(x) = x^3 - 6x^2 + 11x - 6.$$

Check $x = 1$:

$$P(1) = 1 - 6 + 11 - 6 = 0.$$

Therefore

$$x - 1$$

is a factor.

Indeed,

$$P(x) = (x - 1)(x^2 - 5x + 6).$$

Then

Example: Using a Known Root (continued)

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

Hence

$$P(x) = (x - 1)(x - 2)(x - 3).$$

The roots are

$$1, 2, 3.$$

The Remainder Idea

When a polynomial $P(x)$ is divided by $x - a$, the remainder is

$$\boxed{P(a)}.$$

Why?

Write

$$P(x) = (x - a)Q(x) + r,$$

where r is the remainder.

Set $x = a$:

$$P(a) = (a - a)Q(a) + r.$$

Therefore

$$P(a) = r.$$

So

The Remainder Idea (continued)

$$\boxed{\text{remainder} = P(a)}.$$

Remainder Example

Find the remainder when

$$P(x) = 2x^3 - 5x + 7$$

is divided by

$$x - 2.$$

We do not need polynomial long division.
Simply calculate

$$P(2).$$

Thus

$$P(2) = 2(8) - 5(2) + 7 = 16 - 10 + 7 = 13.$$

Therefore

Remainder Example (continued)

13

is the remainder.

Parameter Example

Suppose

$$P(x) = x^3 + kx^2 - 4x + 6$$

has $x - 2$ as a factor.

Then

$$P(2) = 0.$$

Therefore

$$8 + 4k - 8 + 6 = 0.$$

Hence

$$4k + 6 = 0,$$

so

Parameter Example (continued)

$$k = -\frac{3}{2}.$$

TMUA Thinking

A condition expressed in words can often be converted immediately into an algebraic equation.

What Is a Function?

A function is a rule that associates each permitted input with exactly one output.
We may write

$$f : x \mapsto f(x)$$

or simply

$$y = f(x).$$

For example,

$$f(x) = x^2 + 1.$$

Then

$$f(2) = 5,$$

and

What Is a Function? (continued)

$$f(-3) = 10.$$

Key Idea

The symbol f represents the function.

The symbol $f(x)$ represents the output corresponding to input x .

Function Notation Is Not Multiplication

If

$$f(x) = x^2 + 3x,$$

then

$$f(2) = 2^2 + 3(2) = 10.$$

The expression

$$f(2)$$

does not mean

$$f \times 2.$$

It means:

evaluate the function f at input 2.

Function Notation Is Not Multiplication (continued)

Likewise,

$$f(a) = a^2 + 3a.$$

Evaluating More Complicated Inputs

Suppose

$$f(x) = x^2 - 2x + 3.$$

Find

$$f(x + 1).$$

Replace every occurrence of x by $x + 1$:

$$f(x + 1) = (x + 1)^2 - 2(x + 1) + 3.$$

Expanding,

$$= x^2 + 2x + 1 - 2x - 2 + 3.$$

Therefore

$$f(x + 1) = x^2 + 2.$$

A Frequent Error

Suppose again

$$f(x) = x^2 - 2x + 3.$$

It is incorrect to think

$$f(x + 1) = f(x) + 1.$$

In fact,

$$f(x) + 1 = x^2 - 2x + 4,$$

whereas

$$f(x + 1) = x^2 + 2.$$

These are generally different.

A Frequent Error (continued)

Common Trap

Changing the input and changing the output are different operations. This distinction becomes essential when studying graph transformations.

Domain

The **domain** of a function is the set of allowed inputs.
For example,

$$f(x) = \frac{1}{x-3}.$$

We cannot take

$$x = 3$$

because the denominator would be zero.
Therefore the natural real domain is

$$x \in \mathbb{R}, \quad x \neq 3.$$

Another Domain Restriction

Consider

$$f(x) = \sqrt{x-2}.$$

For a real-valued function we require

$$x - 2 \geq 0.$$

Thus

$$x \geq 2.$$

Therefore the domain is

$$\boxed{[2, \infty)}.$$

Another Domain Restriction (continued)

TMUA Thinking

Whenever you see a square root, denominator, logarithm or inverse function, think immediately about possible restrictions.

Range

The **range** is the set of possible outputs.
Consider

$$f(x) = x^2.$$

For real x ,

$$x^2 \geq 0.$$

Therefore

$$\boxed{\text{Range}(f) = [0, \infty)}.$$

Now consider

$$g(x) = x^2 + 5.$$

Since

Range (continued)

$$x^2 + 5 \geq 5,$$

the range is

$$\boxed{[5, \infty)}.$$

Range Through Completing the Square

Find the range of

$$f(x) = x^2 - 6x + 11.$$

Complete the square:

$$f(x) = (x - 3)^2 + 2.$$

Since

$$(x - 3)^2 \geq 0,$$

we have

$$f(x) \geq 2.$$

Thus

$$\boxed{\text{Range}(f) = [2, \infty)}.$$

Again, an algebraic manipulation reveals a graphical property.

Composition of Functions

Suppose

$$f(x) = 2x + 1$$

and

$$g(x) = x^2.$$

The composite function

$$f \circ g$$

means:

$$\boxed{\text{apply } g \text{ first, then apply } f}.$$

Thus

$$(f \circ g)(x) = f(g(x)).$$

Composition of Functions (continued)

Since

$$g(x) = x^2,$$

we obtain

$$f(g(x)) = f(x^2) = 2x^2 + 1.$$

Therefore

$$(f \circ g)(x) = 2x^2 + 1.$$

Order Matters

Using

$$f(x) = 2x + 1, \quad g(x) = x^2,$$

we found

$$(f \circ g)(x) = 2x^2 + 1.$$

Now reverse the order:

$$(g \circ f)(x) = g(f(x)).$$

Thus

$$g(f(x)) = (2x + 1)^2.$$

Hence

$$(g \circ f)(x) = 4x^2 + 4x + 1.$$

Order Matters (continued)

Therefore

$$f \circ g \neq g \circ f$$

in general.

Composition as a Machine

Think of

x

entering one machine and then another:

$$x \longrightarrow g(x) \longrightarrow f(g(x)).$$

For example,

$$x \xrightarrow{g} x^2 \xrightarrow{f} 2x^2 + 1.$$

This representation can make complicated compositions much easier to understand.

Key Idea

Composition is about the order in which operations are performed.

The Idea of an Inverse Function

Suppose

$$f(x) = 2x + 3.$$

The function performs two operations:

$$x \longrightarrow 2x \longrightarrow 2x + 3.$$

To reverse the process:

$$y \longrightarrow y - 3 \longrightarrow \frac{y - 3}{2}.$$

Therefore

$$f^{-1}(x) = \frac{x - 3}{2}.$$

Checking an Inverse

If

$$f(x) = 2x + 3$$

and

$$f^{-1}(x) = \frac{x - 3}{2},$$

then

$$f^{-1}(f(x)) = f^{-1}(2x + 3).$$

Hence

$$= \frac{(2x + 3) - 3}{2} = x.$$

Similarly,

Checking an Inverse (continued)

$$f(f^{-1}(x)) = x.$$

Thus

$$f^{-1} \circ f = f \circ f^{-1} = \text{identity}.$$

A Function Need Not Have an Inverse

Consider

$$f(x) = x^2$$

on all real numbers.

Then

$$f(2) = 4$$

and

$$f(-2) = 4.$$

Two different inputs produce the same output.

Therefore, from the output 4, we cannot uniquely recover the original input.

So $f(x) = x^2$ on $\mathbb{R}$ does not possess an inverse function on its entire natural domain.

Restricting the Domain

If instead we define

$$f(x) = x^2, \quad x \geq 0,$$

then every output corresponds to exactly one input.

The inverse is

$$f^{-1}(x) = \sqrt{x}.$$

If instead the original domain were

$$x \leq 0,$$

then the inverse would be

$$f^{-1}(x) = -\sqrt{x}.$$

Restricting the Domain (continued)

TMUA Thinking

The formula alone does not completely define the behaviour of a function. Domain information can be essential.

Inverse Functions and Graphs

If

$$y = f(x),$$

then exchanging the roles of x and y gives the inverse relationship.
Geometrically, this corresponds to reflection in the line

$$y = x.$$

Thus the graphs of

$$y = f(x)$$

and

$$y = f^{-1}(x)$$

are mirror images in

Inverse Functions and Graphs (continued)

$$y = x.$$

This gives a powerful visual interpretation of inverse functions.

Why Transformations Matter

Suppose you already understand the graph of

$$y = f(x).$$

You should not need to reconstruct from scratch the graphs of

$$y = f(x) + a,$$

$$y = f(x - a),$$

$$y = af(x),$$

or

$$y = f(ax).$$

Instead, recognise each as a transformation of the original graph.

Why Transformations Matter (continued)

Key Idea

Graph transformations turn one known graph into an entire family of related graphs.

Vertical Translation

Compare

$$y = f(x)$$

with

$$y = f(x) + a.$$

Every output increases by a .
Therefore:

$$y = f(x) + a$$

is the graph of $y = f(x)$ translated vertically by a .

If $a > 0$, move upward.

If $a < 0$, move downward.

Example:

Vertical Translation (continued)

$$y = x^2 + 3$$

is $y = x^2$ shifted upward by 3.

Horizontal Translation

Now compare

$$y = f(x)$$

with

$$y = f(x - a).$$

The transformation is

translation a units to the right.

This often feels backwards initially.
For example,

$$y = (x - 4)^2$$

is $y = x^2$ shifted **right** by 4.
Meanwhile,

Horizontal Translation (continued)

$$y = (x + 4)^2$$

is shifted **left** by 4.

Why Does $f(x - a)$ Move Right?

Suppose the original graph contains

$$(x_0, f(x_0)).$$

For the transformed graph

$$y = f(x - a),$$

we need

$$x - a = x_0.$$

Therefore

$$x = x_0 + a.$$

So the point becomes

$$(x_0 + a, f(x_0)).$$

Why Does $f(x - a)$ Move Right? (continued)

Hence every point moves a units to the right.

TMUA Thinking

Understanding the transformation is safer than memorising “inside transformations go backwards.”

Vertical Scaling

Consider

$$y = af(x).$$

Every y -coordinate is multiplied by a .
Thus

$$(x, y) \longrightarrow (x, ay).$$

If

$$|a| > 1,$$

the graph is stretched vertically.
If

$$0 < |a| < 1,$$

it is compressed vertically.

Vertical Scaling (continued)

If

$$a < 0,$$

there is also reflection in the x -axis.

Horizontal Scaling

Consider

$$y = f(ax).$$

If the original graph contains

$$(x_0, f(x_0)),$$

then in the new graph we need

$$ax = x_0.$$

Therefore

$$x = \frac{x_0}{a}.$$

Thus

$$(x_0, y_0) \longrightarrow \left(\frac{x_0}{a}, y_0\right).$$

Horizontal Scaling (continued)

Hence horizontal scale factors behave reciprocally.

Two Important Reflections

Starting from

$$y = f(x),$$

the graph

$$y = -f(x)$$

is a reflection in the x -axis.
Meanwhile,

$$y = f(-x)$$

is a reflection in the y -axis.

These are fundamentally different transformations:

$$f(x) \rightarrow -f(x)$$

Two Important Reflections (continued)

changes outputs,
whereas

$$f(x) \rightarrow f(-x)$$

changes inputs.

Transformation Summary

Starting with $y = f(x)$:

New graph	Transformation
$f(x) + a$	vertical translation
$f(x - a)$	horizontal translation
$af(x)$	vertical scaling
$f(ax)$	horizontal scaling
$-f(x)$	reflection in x-axis
$f(-x)$	reflection in y-axis

Common Trap

Transformations acting **inside** the function affect the input.
Transformations acting **outside** affect the output.

Algebra and Graphs Are Two Languages

Consider

$$f(x) = x^2 - 4x + 3.$$

Factorising gives

$$f(x) = (x - 1)(x - 3),$$

so the roots are

$$1, \quad 3.$$

Completing the square gives

$$f(x) = (x - 2)^2 - 1,$$

so the vertex is

$$(2, -1).$$

Algebra and Graphs Are Two Languages (continued)

The expanded form gives

$$f(0) = 3,$$

so the y -intercept is

$$(0, 3).$$

One Function — Three Pieces of Information

For

$$f(x) = x^2 - 4x + 3,$$

we know:

roots: 1, 3

vertex: (2, -1)

y -intercept: 3.

We can therefore sketch the graph without constructing a table of values.

TMUA Thinking

Graph sketching should usually be based on mathematical structure, not point-by-point plotting.

A Parameter Problem Without Solving

Suppose

$$x^2 - 2kx + k + 3 = 0$$

has two distinct real roots.
Determine the condition on k .
The discriminant must satisfy

$$\Delta > 0.$$

Thus

$$(-2k)^2 - 4(1)(k + 3) > 0.$$

So

$$4k^2 - 4k - 12 > 0.$$

Divide by 4:

A Parameter Problem Without Solving (continued)

$$k^2 - k - 3 > 0.$$

The boundary values satisfy

$$k^2 - k - 3 = 0.$$

Hence

$$k = \frac{1 \pm \sqrt{13}}{2}.$$

Therefore

$$\boxed{k < \frac{1 - \sqrt{13}}{2} \quad \text{or} \quad k > \frac{1 + \sqrt{13}}{2}}.$$

Do Not Automatically Solve for Everything

Suppose the question asks:

How many solutions does $f(x) = g(x)$ have?

You may not need the solutions themselves.

Instead, you might use:

- a discriminant;
- a graph;
- monotonicity;
- sign changes;
- completing the square;
- symmetry;
- inequalities.

Key Idea

The object of problem solving is not to perform as much mathematics as possible. It is to perform the mathematics that answers the question.

TMUA Challenge 1

Let

$$f(x) = x^2 - 4x + 7.$$

Without solving any quadratic equation, determine the minimum value of

$$f(x).$$

Complete the square:

$$f(x) = (x - 2)^2 + 3.$$

Since

$$(x - 2)^2 \geq 0,$$

we obtain

$$\boxed{\min f(x) = 3}.$$

TMUA Challenge 1 (continued)

The minimum occurs at

$$x = 2.$$

TMUA Challenge 2

Suppose

$$f(x) = x^2 + ax + 9$$

has exactly one real root.
What can we deduce about a ?
Exactly one real root means

$$\Delta = 0.$$

Therefore

$$a^2 - 36 = 0.$$

Hence

$$a^2 = 36,$$

so

TMUA Challenge 2 (continued)

$$a = \pm 6.$$

Notice that we never needed to find the root itself.

TMUA Challenge 3

Suppose

$$f(x) = 2x - 1$$

and

$$g(x) = x^2 + 1.$$

Find

$$f(g(2)).$$

First,

$$g(2) = 2^2 + 1 = 5.$$

Then

$$f(5) = 2(5) - 1 = 9.$$

TMUA Challenge 3 (continued)

Therefore

$$\boxed{f(g(2)) = 9}.$$

The order matters:

$$g(f(2))$$

would be a different quantity.

TMUA Challenge 4

Suppose

$$f(x) = \frac{x+1}{x-2}.$$

What values of x are excluded from the domain?

The denominator cannot equal zero:

$$x - 2 \neq 0.$$

Therefore

$$\boxed{x \neq 2}.$$

Now ask a deeper question:

Can $f(x) = 1$?

If

$$\frac{x+1}{x-2} = 1,$$

TMUA Challenge 4 (continued)

then

$$x + 1 = x - 2,$$

which would imply

$$1 = -2.$$

Impossible.

Therefore

1 is not in the range.

TMUA Challenge 5: Structural Recognition

Let

$$f(x) = \frac{x+1}{x-2}.$$

Rewrite:

$$x + 1 = (x - 2) + 3.$$

Hence

$$f(x) = \frac{(x-2)+3}{x-2} = 1 + \frac{3}{x-2}.$$

Now the structure is much clearer.

Immediately we can see:

$$x \neq 2$$

and

TMUA Challenge 5: Structural Recognition (continued)

$$f(x) \neq 1.$$

Thus

$$\text{Domain: } \mathbb{R} \setminus \{2\}$$

and

$$\text{Range: } \mathbb{R} \setminus \{1\}.$$

Why Rewriting Matters

The expressions

$$\frac{x+1}{x-2}$$

and

$$1 + \frac{3}{x-2}$$

represent the same function.

But they reveal different information.

The first form is useful for algebraic substitution.

The second immediately reveals the transformed reciprocal structure

$$y = \frac{1}{x}.$$

Why Rewriting Matters (continued)

TMUA Thinking

One of the most important advanced problem-solving habits is:

Rewrite the object until its structure becomes visible.

Do You Expand or Factor?

Consider

$$(x - 3)(x + 5) = 0.$$

Expanding gives

$$x^2 + 2x - 15 = 0.$$

But this destroys useful information.
The factorised form already tells us

$$x = 3$$

or

$$x = -5.$$

Do You Expand or Factor? (continued)

Key Idea

Algebraic manipulation is not about expanding everything.
Choose the representation that reveals the information you need.

Do You Factor or Complete the Square?

Suppose

$$f(x) = x^2 - 8x + 7.$$

If you need the roots:

$$f(x) = (x - 1)(x - 7).$$

If you need the minimum:

$$f(x) = (x - 4)^2 - 9.$$

If you need the y -intercept:

$$f(x) = x^2 - 8x + 7.$$

Same function.

Different questions.

Different optimal forms.

Representation Is Part of Problem Solving

A mathematical object can often be represented in several ways:

symbolically $\leftrightarrow$ graphically $\leftrightarrow$ numerically $\leftrightarrow$ geometrically

A difficult problem in one representation may become simple after changing representation.

For example:

$$x^2 - 6x + 13$$

may look awkward.

But

$$(x - 3)^2 + 4$$

immediately reveals that it is always positive.

Part 2: Core Ideas

You should now be comfortable with:

- interpreting equations as conditions;
- simultaneous equations as intersections;
- using discriminants without necessarily solving;
- solving linear and quadratic inequalities;
- recognising polynomial roots and factors;
- using polynomial remainders efficiently;
- understanding function notation;
- determining domain and range;
- composing functions;
- constructing inverse functions;
- understanding domain restrictions for inverses;
- recognising graph transformations.

The Most Important Habits

- 1 Read structure before calculating.
- 2 Ask what information the question actually requires.
- 3 Choose the most informative algebraic form.
- 4 Move freely between equations and graphs.
- 5 Check domains and restrictions.
- 6 Treat parameters as information, not obstacles.
- 7 Avoid irreversible algebraic steps unless you check the resulting candidates.

Final Self-Check

Before continuing, make sure you can explain:

Questions to Ask Yourself

- 1 Why does $f(x) = g(x)$ represent graph intersections?
- 2 Why does the sign of the discriminant determine the number of intersections in a linear-quadratic system?
- 3 Why must an inequality reverse when dividing by a negative number?
- 4 Why does $P(a) = 0$ imply that $x - a$ is a factor?
- 5 What is the difference between $f(g(x))$ and $g(f(x))$?
- 6 Why does x^2 need a restricted domain to have an inverse?
- 7 Why does $f(x - a)$ shift a graph to the right?
- 8 When should you factor, expand, or complete the square?

From Algebra to Sequences

Algebra has given us the machinery to describe relationships.
The next natural question is:

What happens when mathematical quantities evolve according to a pattern?

This leads us to

Sequences and Series

where we will study:

- sequence notation;
- arithmetic progressions;
- geometric progressions;
- finite sums;

From Algebra to Sequences (continued)

- infinite geometric series;
- recurrence relations;
- structural sequence problems.

Do Not Manipulate Algebra Blindly.

Every transformation should have a purpose.

Recognise $\rightarrow$ Represent $\rightarrow$ Reason $\rightarrow$ Solve

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 3: Sequences and Series

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From Algebra to Sequences

In Parts 1 and 2, we developed algebraic tools for studying mathematical relationships. Now we ask a different question:

What happens when mathematical quantities evolve according to a pattern?

Examples:

$$2, 5, 8, 11, 14, \dots$$

$$3, 6, 12, 24, 48, \dots$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

These lead naturally to the study of

From Algebra to Sequences (continued)

Sequences and Series.

Part 3 Roadmap

We will study:

- ① Sequences, series and progressions
- ② Explicit formulae
- ③ Recurrence relations
- ④ Pattern recognition
- ⑤ Periodic behaviour
- ⑥ Sigma notation
- ⑦ Arithmetic progressions
- ⑧ Arithmetic series
- ⑨ Geometric progressions
- ⑩ Finite geometric series
- ⑪ Infinite geometric series
- ⑫ Convergence and $|r| < 1$
- ⑬ Partial sums
- ⑭ Extracting part of a progression
- ⑮ Derived sequences
- ⑯ TMUA-style structural problems

Three Important Words

We should distinguish three related ideas.

Sequence

A **sequence** is an ordered list of terms.

$$u_1, u_2, u_3, \dots$$

Series

A **series** is obtained by adding the terms of a sequence.

$$u_1 + u_2 + u_3 + \dots$$

The word **progression** is often used for structured sequences or series such as arithmetic and geometric progressions.

Order Matters

Consider

$$1, 2, 3, 4, \dots$$

and

$$4, 3, 2, 1, \dots$$

These contain some of the same numbers, but they are not the same sequence.

Why?

Because a sequence is **ordered**.

We usually write

$$u_n$$

for the n -th term.

Thus

Order Matters (continued)

$$u_1 = \text{first term,}$$

$$u_2 = \text{second term,}$$

and so on.

Explicit Formulae

A sequence may be defined directly by giving a formula for the n -th term.
For example,

$$u_n = 3n + 2.$$

Then

$$u_1 = 5,$$

$$u_2 = 8,$$

$$u_3 = 11,$$

$$u_4 = 14.$$

Hence the sequence is

Explicit Formulae (continued)

5, 8, 11, 14, ...

Key Idea

An explicit formula allows us to calculate u_n directly from n .

Finding a Distant Term

Suppose

$$u_n = 5n - 7.$$

Find u_{100} .

Simply substitute $n = 100$:

$$u_{100} = 5(100) - 7.$$

Therefore

$$u_{100} = 493.$$

We did not need to calculate

$$u_1, u_2, \dots, u_{99}.$$

That is the advantage of an explicit formula.

What Is a Recurrence Relation?

Instead of giving u_n directly, we may define a term using previous terms.

For example,

$$u_1 = 3,$$

and

$$u_{n+1} = u_n + 4.$$

Then

$$u_2 = 7,$$

$$u_3 = 11,$$

$$u_4 = 15.$$

Thus

What Is a Recurrence Relation? (continued)

3, 7, 11, 15, ...

Key Idea

A recurrence tells us how the sequence evolves from one term to another.

Explicit vs Recursive Description

The sequence

3, 7, 11, 15, ...

can be described recursively by

$$u_1 = 3, \quad u_{n+1} = u_n + 4.$$

It can also be described explicitly:

$$u_n = 3 + 4(n - 1).$$

Equivalently,

$$u_n = 4n - 1.$$

These are two different descriptions of the same sequence.

Why Recurrences Can Be More Subtle

Consider

$$x_0 = 1, \quad x_1 = 2,$$

and

$$x_{n+2} = |x_n - x_{n+1}|.$$

Calculate carefully:

$$x_2 = |1 - 2| = 1,$$

$$x_3 = |2 - 1| = 1,$$

$$x_4 = |1 - 1| = 0,$$

$$x_5 = |1 - 0| = 1,$$

Why Recurrences Can Be More Subtle (continued)

$$x_6 = |0 - 1| = 1,$$

$$x_7 = |1 - 1| = 0.$$

Thus the beginning is

$$1, 2, 1, 1, 0, 1, 1, 0, 1, 1, 0, \dots$$

Do Not Guess the Pattern Too Early

After seeing

$$1, 2, 1, 1,$$

we might be tempted to guess a pattern.
But the recurrence

$$x_{n+2} = |x_n - x_{n+1}|$$

depends on **two previous terms**.

Therefore a convincing repeating structure requires us to examine enough consecutive terms to understand the state of the recurrence.

Common Trap

A short numerical pattern is evidence, not proof.

Always ask whether the recurrence itself forces the pattern to continue.

A General Pattern-Recognition Principle

Suppose a recurrence depends on

$$u_n, \quad u_{n+1}, \quad \dots, \quad u_{n+k-1}.$$

Then the “state” of the recurrence is determined by a block of k consecutive terms.

If exactly the same state occurs again, the future evolution will repeat in the same way.

TMUA Thinking

When analysing recurrence-generated sequences, do not only look for repeated individual terms.
Look for repeated **states**.

Why Do We Need Sigma Notation?

Writing

$$u_1 + u_2 + u_3 + \cdots + u_{100}$$

is manageable.

But longer or more complicated sums quickly become cumbersome.

We therefore write

$$\sum_{k=1}^{100} u_k$$

where

$$\Sigma$$

means “sum.”

The lower and upper limits tell us which values of the index are included.

Reading Sigma Notation

Consider

$$\sum_{k=1}^5 (2k + 1).$$

This means

$$(2(1) + 1) + (2(2) + 1) + (2(3) + 1) + (2(4) + 1) + (2(5) + 1).$$

Thus

$$= 3 + 5 + 7 + 9 + 11.$$

Therefore

$$\sum_{k=1}^5 (2k + 1) = 35.$$

How Many Terms Are There?

Consider

$$\sum_{k=m}^n u_k.$$

How many terms are included?

They are

$$u_m, u_{m+1}, \dots, u_n.$$

The number of terms is

$$\boxed{n - m + 1}.$$

For example,

$$u_4 + u_5 + \dots + u_{10}$$

How Many Terms Are There? (continued)

contains

$$10 - 4 + 1 = 7$$

terms.

The Fence-Post Error

A very common mistake is to say that

$$u_m + u_{m+1} + \cdots + u_n$$

contains

$$n - m$$

terms.

It does not.

It contains

$$\boxed{n - m + 1}.$$

For example, the integers

$$4, 5, 6, 7$$

contain four numbers, even though

The Fence-Post Error (continued)

$$7 - 4 = 3.$$

Common Trap

When both endpoints are included, remember the extra $+1$.

A Useful Counting Check

How many terms are in

$$\sum_{n=0}^{100} x_n?$$

The terms are

$$x_0, x_1, x_2, \dots, x_{100}.$$

Therefore the number of terms is

$$100 - 0 + 1.$$

Hence

$$\boxed{101}.$$

This small detail can completely change the answer in a pattern-based sum.

Arithmetic Progressions

An arithmetic progression has a constant difference between successive terms.
For example,

$$5, 8, 11, 14, 17, \dots$$

has common difference

$$d = 3.$$

If the first term is a , then the sequence is

$$a, a + d, a + 2d, a + 3d, \dots$$

Therefore

$$\boxed{u_n = a + (n - 1)d}.$$

Why $n - 1$?

Suppose the first term is a .

To reach the second term, we add d once:

$$u_2 = a + d.$$

To reach the third term, we add d twice:

$$u_3 = a + 2d.$$

To reach the n -th term, we add d

$$n - 1$$

times.

Hence

$$u_n = a + (n - 1)d.$$

Why $n - 1$? (continued)

Common Trap

The formula is not $a + nd$.

At $n = 1$, the first term must still equal a .

Recognising an Arithmetic Sequence

A sequence is arithmetic precisely when

$$u_{n+1} - u_n$$

is constant.

Thus

$$\boxed{u_{n+1} - u_n = d}.$$

For example, if

$$u_n = 7 - 3n,$$

then

$$u_{n+1} = 7 - 3(n + 1) = 4 - 3n.$$

Therefore

Recognising an Arithmetic Sequence (continued)

$$u_{n+1} - u_n = (4 - 3n) - (7 - 3n) = -3.$$

Hence the sequence is arithmetic with

$$\boxed{d = -3}.$$

Finding the n -th Term

Suppose an AP begins

$$11, 17, 23, 29, \dots$$

Then

$$a = 11, \quad d = 6.$$

Therefore

$$u_n = 11 + (n - 1)6.$$

Simplifying,

$$u_n = 11 + 6n - 6.$$

Hence

$$\boxed{u_n = 6n + 5}.$$

TMUA-Style AP Question

The 7-th term of an arithmetic progression is 25, and the 13-th term is 43.

Find the common difference.

We have

$$u_7 = a + 6d = 25$$

and

$$u_{13} = a + 12d = 43.$$

Subtract:

$$6d = 18.$$

Therefore

$$\boxed{d = 3}.$$

Then

TMUA-Style AP Question (continued)

$$a + 18 = 25,$$

so

$$\boxed{a = 7}.$$

From an AP to an Arithmetic Series

Suppose

$$u_n = a + (n - 1)d.$$

The sum of the first n terms is

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d].$$

The standard result is

$$\boxed{S_n = \frac{n}{2} [2a + (n - 1)d]}.$$

Equivalently,

$$\boxed{S_n = \frac{n}{2} (u_1 + u_n)}.$$

Why Does the AP Sum Formula Work?

Write

$$S_n = a + (a + d) + \cdots + [a + (n - 1)d].$$

Now reverse the order:

$$S_n = [a + (n - 1)d] + \cdots + (a + d) + a.$$

Add vertically.
Every pair gives

$$2a + (n - 1)d.$$

There are n such pairs, so

$$2S_n = n[2a + (n - 1)d].$$

Therefore

Why Does the AP Sum Formula Work? (continued)

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

A More Intuitive Interpretation

For an arithmetic progression,

$$S_n = n \times \frac{\text{first term} + \text{last term}}{2}.$$

That is,

$$\text{sum} = \text{number of terms} \times \text{average term}.$$

Why?

Because in an arithmetic progression the terms are distributed symmetrically around their average. This interpretation is often more useful than memorising the formula mechanically.

Sum of the First n Natural Numbers

Take

$$1 + 2 + 3 + \cdots + n.$$

This is an AP with

$$a = 1, \quad d = 1, \quad u_n = n.$$

Therefore

$$S_n = \frac{n}{2}(1 + n).$$

Hence

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example: Sum of an AP

Find

$$7 + 11 + 15 + \cdots + 99.$$

First determine the number of terms.

$$99 = 7 + (n - 1)4.$$

Thus

$$92 = 4(n - 1),$$

so

$$n - 1 = 23$$

and

$$n = 24.$$

Example: Sum of an AP (continued)

Therefore

$$S_{24} = \frac{24}{2}(7 + 99).$$

Hence

$$S_{24} = 12(106).$$

Therefore

$$\boxed{S_{24} = 1272}.$$

Adding Arithmetic Sequences

Suppose

$$u_n = a + (n - 1)d$$

and

$$v_n = A + (n - 1)D.$$

Then

$$u_n + v_n = (a + A) + (n - 1)(d + D).$$

Therefore the new sequence

$$w_n = u_n + v_n$$

is also arithmetic.

Its first term is

Adding Arithmetic Sequences (continued)

$$a + A$$

and its common difference is

$$d + D.$$

A More General Observation

Consider

$$w_n = \alpha u_n + \beta v_n,$$

where u_n and v_n are arithmetic.

Then

$$w_{n+1} - w_n = \alpha(u_{n+1} - u_n) + \beta(v_{n+1} - v_n).$$

Since

$$u_{n+1} - u_n = d$$

and

$$v_{n+1} - v_n = D,$$

we obtain

A More General Observation (continued)

$$w_{n+1} - w_n = \alpha d + \beta D,$$

which is constant.

Thus w_n is also arithmetic.

Geometric Progressions

A geometric progression has a constant ratio between successive terms.
For example,

$$3, 6, 12, 24, 48, \dots$$

has common ratio

$$r = 2.$$

If the first term is a , then

$$a, ar, ar^2, ar^3, \dots$$

and therefore

$$u_n = ar^{n-1}.$$

Recognising a GP

For a geometric progression,

$$\frac{u_{n+1}}{u_n} = r$$

whenever the ratio is defined.
Thus

$$\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = r.$$

Compare this with an AP:

$$u_{n+1} - u_n = d.$$

So:

AP $\longrightarrow$ constant difference,

Recognising a GP (continued)

GP $\longrightarrow$ constant ratio.

Finding a GP Formula

Consider

5, 15, 45, 135, ...

The first term is

$$a = 5.$$

The common ratio is

$$r = 3.$$

Therefore

$$u_n = 5 \cdot 3^{n-1}.$$

For example,

$$u_6 = 5 \cdot 3^5 = 1215.$$

Negative Common Ratio

Consider

$$8, -4, 2, -1, \frac{1}{2}, \dots$$

The common ratio is

$$r = -\frac{1}{2}.$$

Thus

$$u_n = 8 \left(-\frac{1}{2} \right)^{n-1}.$$

The signs alternate because the ratio is negative.
At the same time,

$$|u_n|$$

Negative Common Ratio (continued)

becomes progressively smaller because

$$\left| -\frac{1}{2} \right| < 1.$$

This will matter when we study infinite series.

The Finite Geometric Series

Consider

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiply by r :

$$rS_n = ar + ar^2 + \cdots + ar^n.$$

Subtract:

$$S_n - rS_n = a - ar^n.$$

Therefore

$$S_n(1 - r) = a(1 - r^n).$$

Hence

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

The Finite Geometric Series (continued)

for $r \neq 1$.

Equivalent Form of the GP Sum

We have

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Multiplying numerator and denominator by -1 ,

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Thus

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}.$$

Both forms are equivalent.
Choose whichever is more convenient.

Example: Finite GP

Find

$$3 + 6 + 12 + \cdots + 384.$$

Here

$$a = 3, \quad r = 2.$$

First find the number of terms:

$$3 \cdot 2^{n-1} = 384.$$

Thus

$$2^{n-1} = 128 = 2^7.$$

Therefore

$$n = 8.$$

Example: Finite GP (continued)

Now

$$S_8 = \frac{3(2^8 - 1)}{2 - 1}.$$

Hence

$$S_8 = 765.$$

Can an Infinite Sum Be Finite?

Consider

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

We keep adding positive numbers forever.

Yet the total approaches

2.

This seems surprising at first.

The key is that the added terms become smaller and smaller.

The partial sums are

$$1, \quad \frac{3}{2}, \quad \frac{7}{4}, \quad \frac{15}{8}, \dots$$

and these approach 2.

From Finite Sum to Infinite Sum

For a GP,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

If

$$|r| < 1,$$

then

$$r^n \longrightarrow 0$$

as

$$n \longrightarrow \infty.$$

Therefore

$$S_n \longrightarrow \frac{a}{1 - r}.$$

From Finite Sum to Infinite Sum (continued)

Hence

$$S_{\infty} = \frac{a}{1 - r}$$

provided

$$|r| < 1.$$

Why Is $|r| < 1$ Essential?

For an infinite GP to converge, its terms must approach zero.
Since

$$u_n = ar^{n-1},$$

we require

$$r^{n-1} \longrightarrow 0.$$

This happens when

$$\boxed{|r| < 1}.$$

If

$$r = 2,$$

the terms grow.

If

Why Is $|r| < 1$ Essential? (continued)

$$r = 1,$$

they remain constant.

If

$$r = -1,$$

they alternate without shrinking.

Therefore the infinite sum does not converge in these cases.

Example: Infinite GP

Find

$$12 + 6 + 3 + \frac{3}{2} + \dots.$$

Here

$$a = 12,$$

and

$$r = \frac{1}{2}.$$

Since

$$\left| \frac{1}{2} \right| < 1,$$

the series converges.

Example: Infinite GP (continued)

Therefore

$$S_{\infty} = \frac{12}{1 - \frac{1}{2}}.$$

Hence

$$S_{\infty} = \frac{12}{1/2} = 24.$$

Thus

$$\boxed{S_{\infty} = 24}.$$

Example With an Alternating GP

Evaluate

$$10 - 5 + \frac{5}{2} - \frac{5}{4} + \dots$$

Here

$$a = 10$$

and

$$r = -\frac{1}{2}.$$

Since

$$|r| = \frac{1}{2} < 1,$$

the series converges.

Thus

Example With an Alternating GP (continued)

$$S_{\infty} = \frac{10}{1 - (-1/2)} = \frac{10}{3/2}.$$

Therefore

$$\boxed{S_{\infty} = \frac{20}{3}}.$$

A Common Convergence Mistake

Suppose

$$r = -\frac{3}{2}.$$

Someone might say:

$$r < 1,$$

so the infinite series converges.

This is incorrect.

We need

$$|r| < 1,$$

not merely

$$r < 1.$$

A Common Convergence Mistake (continued)

Here

$$\left| -\frac{3}{2} \right| = \frac{3}{2} > 1.$$

So the terms grow in magnitude and the series does not converge.

Writing a GP With Sigma Notation

The finite GP

$$a + ar + ar^2 + \dots + ar^{n-1}$$

can be written as

$$\sum_{k=0}^{n-1} ar^k.$$

Equally, we may write

$$\sum_{k=1}^n ar^{k-1}.$$

These are exactly the same sum.

The indexing has changed, but the terms have not.

Check the First and Last Terms

Whenever sigma notation is re-indexed, check:

- ❶ the first term;
- ❷ the last term;
- ❸ the number of terms.

For

$$\sum_{k=0}^{n-1} ar^k,$$

the first term is

$$ar^0 = a,$$

and the last is

$$ar^{n-1}.$$

Check the First and Last Terms (continued)

There are

$$(n - 1) - 0 + 1 = n$$

terms.

TMUA Thinking

Endpoint checks prevent many indexing errors.

Summing Part of a GP

Suppose we want

$$ar^m + ar^{m+1} + \dots + ar^n, \quad n \geq m.$$

How many terms are there?

The exponents are

$$m, m + 1, \dots, n.$$

Therefore the number of terms is

$$\boxed{n - m + 1}.$$

Factor out the first term:

$$ar^m (1 + r + r^2 + \dots + r^{n-m}).$$

This is now an ordinary finite GP.

Method 1: Treat It as a New GP

The sum

$$ar^m + ar^{m+1} + \cdots + ar^n$$

has

$$\text{first term} = ar^m,$$

$$\text{common ratio} = r,$$

and

$$\text{number of terms} = n - m + 1.$$

Therefore

$$S = ar^m \frac{1 - r^{n-m+1}}{1 - r}.$$

Method 1: Treat It as a New GP (continued)

Hence

$$S = \frac{ar^m(1 - r^{n-m+1})}{1 - r}.$$

Method 2: Difference of Partial Sums

Define

$$S_k = a + ar + \cdots + ar^{k-1}.$$

Then

$$ar^m + ar^{m+1} + \cdots + ar^n$$

is obtained by taking the first $n + 1$ terms and removing the first m terms.
Therefore

$$ar^m + \cdots + ar^n = S_{n+1} - S_m.$$

Common Trap

It is not $S_n - S_m$.

The final exponent n corresponds to term number $n + 1$ when the first exponent is 0.

Why S_{n+1} ?

Consider

$$a + ar + ar^2 + \cdots + ar^n.$$

Count the exponents:

$$0, 1, 2, \dots, n.$$

There are

$$n + 1$$

terms.

Therefore

$$a + ar + \cdots + ar^n = S_{n+1}.$$

This is another version of the fence-post issue.

Why S_{n+1} ? (continued)

TMUA Thinking

Whenever indices begin at zero, be particularly careful when translating between exponents and term numbers.

Squaring Every Term of a GP

Suppose

$$u_n = ar^{n-1}.$$

Then

$$u_n^2 = a^2 r^{2(n-1)}.$$

Thus the squared sequence is

$$a^2, a^2 r^2, a^2 r^4, \dots$$

This is itself a GP with

first term = a^2

and

common ratio = r^2 .

Raising Every Term to a Power

More generally, suppose

$$u_n = ar^{n-1}.$$

Raise each term to power k :

$$u_n^k = a^k r^{k(n-1)}.$$

Therefore

$$u_n^k = a^k (r^k)^{n-1}.$$

Hence the new sequence is geometric with

$$\text{first term} = a^k$$

and

$$\text{ratio} = r^k.$$

Infinite Sum of the Squared Terms

Suppose

$$u_n = ar^{n-1}$$

and

$$|r| < 1.$$

Then

$$u_n^2 = a^2 (r^2)^{n-1}.$$

Since

$$|r| < 1 \implies r^2 < 1,$$

the squared terms form a convergent GP.

Therefore

Infinite Sum of the Squared Terms (continued)

$$\sum_{n=1}^{\infty} u_n^2 = \frac{a^2}{1-r^2}.$$

This type of transformation is useful in more advanced sequence problems.

TMUA Challenge 1

An arithmetic progression satisfies

$$u_5 = 17, \quad u_{11} = 41.$$

Find u_{20} .

We have

$$a + 4d = 17$$

and

$$a + 10d = 41.$$

Subtract:

$$6d = 24,$$

so

TMUA Challenge 1 (continued)

$$d = 4.$$

Then

$$a = 1.$$

Therefore

$$u_{20} = 1 + 19(4) = \boxed{77}.$$

TMUA Challenge 2

The first three terms of an arithmetic progression are

$$x, \quad 2x + 1, \quad 4x - 5.$$

Find x .

For an AP,

$$(2x + 1) - x = (4x - 5) - (2x + 1).$$

Thus

$$x + 1 = 2x - 6.$$

Therefore

$$\boxed{x = 7}.$$

The progression begins

TMUA Challenge 2 (continued)

7, 15, 23, ...

TMUA Challenge 3

The first three terms of a geometric progression are

x , 6, 12.

Find x .

For a GP,

$$\frac{6}{x} = \frac{12}{6}.$$

Therefore

$$\frac{6}{x} = 2.$$

Hence

$$\boxed{x = 3}.$$

Alternatively, for three consecutive nonzero GP terms,

TMUA Challenge 3 (continued)

$$\boxed{(\text{middle})^2 = (\text{first})(\text{third})}.$$

Thus

$$36 = 12x,$$

again giving $x = 3$.

A Useful Three-Term GP Property

If

$$a, b, c$$

are consecutive terms of a GP, then

$$\frac{b}{a} = \frac{c}{b}.$$

Cross-multiplying,

$$b^2 = ac.$$

Therefore

$$\boxed{b^2 = ac}.$$

This means the middle term is a geometric mean of its neighbours in the appropriate setting. Compare with an AP:

A Useful Three-Term GP Property (continued)

$$2b = a + c.$$

Thus:

AP: middle = arithmetic mean,

GP: middle = geometric mean.

TMUA Challenge 4

A convergent geometric series has first term 8 and sum to infinity 12.

Find the common ratio.

Using

$$S_{\infty} = \frac{a}{1-r},$$

we have

$$12 = \frac{8}{1-r}.$$

Thus

$$12(1-r) = 8.$$

So

$$1-r = \frac{2}{3}.$$

TMUA Challenge 4 (continued)

Therefore

$$r = \frac{1}{3}.$$

TMUA Challenge 5

A geometric series has

$$S_{\infty} = 20$$

and first term

$$a = 5.$$

Find r .

We have

$$20 = \frac{5}{1-r}.$$

Thus

$$1-r = \frac{1}{4},$$

and therefore

TMUA Challenge 5 (continued)

$$r = \frac{3}{4}.$$

Check convergence:

$$\left| \frac{3}{4} \right| < 1.$$

So the result is valid.

TMUA Challenge 6: Ratio Reasoning

Suppose a GP has positive terms and

$$u_3 = 12, \quad u_5 = 48.$$

Find the common ratio.

Since

$$u_3 = ar^2$$

and

$$u_5 = ar^4,$$

divide:

$$\frac{u_5}{u_3} = r^2.$$

Therefore

TMUA Challenge 6: Ratio Reasoning (continued)

$$\frac{48}{12} = r^2.$$

Thus

$$r^2 = 4.$$

Because the terms are positive,

$$\boxed{r = 2}.$$

Why the Positivity Condition Matters

From

$$r^2 = 4,$$

algebra alone gives

$$r = \pm 2.$$

If the problem states that every term is positive, then a negative ratio is impossible because the signs would alternate.

Therefore

$$r = 2.$$

Why the Positivity Condition Matters (continued)

TMUA Thinking

Never ignore qualitative information such as:

- positive;
- increasing;
- decreasing;
- integer;
- convergent.

Such information can eliminate algebraic possibilities.

TMUA Challenge 7: Recurrence

Let

$$x_1 = 2$$

and

$$x_{n+1} = 3 - x_n.$$

Find the first several terms.

We have

$$x_1 = 2,$$

$$x_2 = 1,$$

$$x_3 = 2,$$

TMUA Challenge 7: Recurrence (continued)

$$x_4 = 1.$$

Therefore

$$2, 1, 2, 1, \dots$$

The sequence has period 2.
Hence

$$x_n = \begin{cases} 2, & n \text{ odd,} \\ 1, & n \text{ even.} \end{cases}$$

Using Periodicity

For the sequence

$$2, 1, 2, 1, \dots,$$

suppose we want

$$x_{2026}.$$

Since 2026 is even,

$$x_{2026} = 1.$$

No repeated computation is necessary.

Similarly,

$$x_{2027} = 2.$$

Using Periodicity (continued)

Key Idea

Once a recurrence is shown to be periodic, very distant terms can often be found using modular arithmetic.

Periodicity and Remainders

Suppose a sequence repeats every 5 terms:

$$a, b, c, d, e, a, b, c, d, e, \dots$$

To find the 1003-rd term, calculate

$$1003 \pmod{5}.$$

Since

$$1003 = 5(200) + 3,$$

the 1003-rd term occupies position 3 in the repeating block.
Therefore it is

$$\boxed{c}.$$

Sequence Problems Are Often Recognition Problems

When you see a sequence, ask:

- ❶ Is the difference constant?
- ❷ Is the ratio constant?
- ❸ Is the pattern periodic?
- ❹ Is there a recurrence?
- ❺ Can I derive an explicit formula?
- ❻ Do I need a term or a sum?
- ❼ How many terms are actually present?
- ❽ Can I exploit symmetry or grouping?

TMUA Thinking

Do not decide “AP” or “GP” simply because the first few terms look suggestive. Verify the defining structure.

Term or Sum?

A very common source of errors is confusing

$$u_n$$

with

$$S_n.$$

Remember:

$$u_n = n\text{-th term}$$

whereas

$$S_n = u_1 + u_2 + \cdots + u_n.$$

For an AP,

$$u_n = a + (n - 1)d,$$

Term or Sum? (continued)

but

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

For a GP,

$$u_n = ar^{n-1},$$

but

$$S_n = \frac{a(1-r^n)}{1-r}.$$

Formula Selection

Before inserting numbers into a formula, identify what the question is asking.

Question	Useful quantity
Find the n -th AP term	$u_n = a + (n-1)d$
Sum first n AP terms	$S_n = \frac{n}{2}[2a + (n-1)d]$
Find the n -th GP term	$u_n = ar^{n-1}$
Sum first n GP terms	$S_n = \frac{a(1-r^n)}{1-r}$
Infinite GP	$S_\infty = \frac{a}{1-r}$

But Do Not Become Formula-Dependent

Formulae are useful, but they should not replace understanding.
For example, if asked to find

$$1 + 2 + 3 + \cdots + 100,$$

you can use

$$\frac{100(101)}{2}.$$

But understanding why the formula works helps you recognise related structures.
Similarly, understanding the derivation of

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

makes it easier to manipulate unusual geometric sums.

But Do Not Become Formula-Dependent (continued)

Key Idea

Know the formula.
Understand the formula.
Recognise when the formula needs to be adapted.

A Mixed AP Problem

Suppose an arithmetic progression has

$$u_3 + u_7 = 30.$$

What can we deduce?

Since

$$u_3 = a + 2d$$

and

$$u_7 = a + 6d,$$

we have

$$2a + 8d = 30.$$

Therefore

A Mixed AP Problem (continued)

$$a + 4d = 15.$$

But

$$a + 4d = u_5.$$

Hence

$$\boxed{u_5 = 15}.$$

No need to determine a and d individually.

Why Did That Work?

In an AP,

$$u_3, u_5, u_7$$

are equally spaced terms.
Therefore

$$u_5$$

is the arithmetic mean of u_3 and u_7 .
Thus

$$u_5 = \frac{u_3 + u_7}{2}.$$

Since

$$u_3 + u_7 = 30,$$

Why Did That Work? (continued)

we immediately get

$$u_5 = 15.$$

TMUA Thinking

This is a good example of structural reasoning replacing unnecessary simultaneous equations.

General AP Symmetry

For an arithmetic progression,

$$u_{m-k}$$

and

$$u_{m+k}$$

are equally far from u_m .

Therefore

$$u_{m-k} + u_{m+k} = 2u_m.$$

For example,

$$u_2 + u_8 = 2u_5,$$

$$u_{10} + u_{20} = 2u_{15}.$$

This symmetry is extremely useful in problems involving sums of selected AP terms.

The GP Analogue

For a GP,

$$u_{m-k}$$

and

$$u_{m+k}$$

satisfy

$$u_{m-k}u_{m+k} = u_m^2.$$

Therefore

$$u_m^2 = u_{m-k}u_{m+k}.$$

Compare:

$$\text{AP: } 2u_m = u_{m-k} + u_{m+k}$$

The GP Analogue (continued)

with

$$\text{GP: } u_m^2 = u_{m-k} u_{m+k}.$$

This is the arithmetic-mean/geometric-mean structure appearing naturally in sequences.

Part 3: What You Should Now Know

You should now be comfortable with:

- sequence and series terminology;
- explicit formulae;
- recurrence relations;
- recognising periodic patterns;
- sigma notation;
- counting indexed terms correctly;
- arithmetic progressions;
- arithmetic sums;
- geometric progressions;
- finite geometric sums;
- infinite geometric sums;
- the convergence condition $|r| < 1$;
- partial geometric sums;
- derived geometric progressions;
- AP and GP symmetry.

The Formula Map

$$\text{AP: } u_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\text{GP: } u_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r}, \quad |r| < 1$$

The Recognition Map

$$u_{n+1} - u_n = \text{constant}$$



Arithmetic Progression

The Recognition Map (continued)

$$\frac{u_{n+1}}{u_n} = \text{constant}$$



Geometric Progression

The Most Important Sequence Habits

- ❶ Write enough terms before claiming a pattern.
- ❷ Distinguish a term u_n from a sum S_n .
- ❸ Count indexed terms carefully.
- ❹ Check whether the sequence is really arithmetic or geometric.
- ❺ For recurrences, identify how much previous information determines the next state.
- ❻ Check convergence before using S_∞ .
- ❼ Use symmetry instead of unnecessary algebra where possible.
- ❽ When using sigma notation, inspect both endpoints.

Part 3 Self-Check

Can You Explain These Without Looking Back?

- ❶ What is the difference between a sequence and a series?
- ❷ Why does an AP have $u_n = a + (n - 1)d$?
- ❸ Why does $S_n = \frac{n}{2}(u_1 + u_n)$ for an AP?
- ❹ Why does a GP have $u_n = ar^{n-1}$?
- ❺ How is the finite GP formula derived?
- ❻ Why is $|r| < 1$ required for an infinite GP?
- ❼ Why are there $n - m + 1$ terms from index m to n ?
- ❽ Why is $u_{m-k} + u_{m+k} = 2u_m$ for an AP?
- ❾ What is the corresponding identity for a GP?

How to Practise This Topic

Do not practise only questions of the form:

“Find S_{20} .”

Also practise questions where you must first determine:

- whether the sequence is AP or GP;
- which information is actually relevant;
- how many terms are present;
- whether a series converges;
- whether a recurrence becomes periodic;
- whether a distant term can be found without generating every preceding term;
- whether a sum can be reorganised or simplified structurally.

TMUA Thinking

The goal is recognition and reasoning, not merely substitution into standard formulae.

Coming in Part 4

TMUA Mathematics Masterclass

Part 4: Coordinate Geometry in the (x, y) -Plane

We will study:

- coordinates and geometric interpretation;
- distance between points;
- midpoint and section ideas;
- gradients;
- equations of straight lines;
- parallel and perpendicular lines;
- intersections;
- circles;
- equations of circles;

Coming in Part 4 (continued)

- tangency;
- coordinate transformations;
- algebra–geometry connections;
- TMUA-style coordinate problems.

Do Not Just Spot Patterns.

Understand why the pattern must continue.

Generate → Recognise → Justify → Exploit

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 4: Coordinate Geometry in the (x, y) -Plane

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Algebra Meets Geometry

Coordinate geometry creates a bridge between

Algebra and Geometry.

An equation such as

$$y = 2x + 3$$

is simultaneously

- an algebraic relationship between x and y ;
- a geometric object: a straight line.

Algebra Meets Geometry (continued)

Similarly,

$$x^2 + y^2 = 25$$

is an equation and also describes a circle.

Key Idea

Coordinate geometry allows us to solve geometric problems using algebra and algebraic problems using geometry.

Part 4 Roadmap

We will study:

- ① Coordinates and straight lines
- ② Gradient
- ③ Gradient from two points
- ④ $y = mx + c$
- ⑤ Point-gradient form
- ⑥ General form $ax + by + c = 0$
- ⑦ Finding equations of lines
- ⑧ Parallel lines
- ⑨ Perpendicular lines
- ⑩ Horizontal and vertical lines
- ⑪ Intersections and simultaneous equations
- ⑫ Equations of circles
- ⑬ Centre-radius form

Part 4 Roadmap (continued)

- ⑭ Expanded circle equations
- ⑮ Completing the square
- ⑯ Recognising when no real circle exists
- ⑰ Line-circle intersections
- ⑱ TMUA structural problems

The Equation $y = mx + c$

One of the most important equations in coordinate geometry is

$$y = mx + c.$$

Here

$$m = \text{gradient}$$

and

$$c = \text{y-intercept.}$$

For example,

$$y = 3x + 5$$

has

$$m = 3, \quad c = 5.$$

The Equation $y = mx + c$ (continued)

Therefore the line crosses the y -axis at

$$(0, 5).$$

What Does Gradient Mean?

The gradient measures how y changes relative to x .

$$m = \frac{\text{change in } y}{\text{change in } x}$$

If

$$m = 2,$$

then an increase of 1 in x produces an increase of 2 in y .

If

$$m = -3,$$

then an increase of 1 in x produces a decrease of 3 in y .

What Does Gradient Mean? (continued)

Key Idea

Gradient is not merely “steepness.”
It is a **rate of change**.

Positive and Negative Gradients

If

$$m > 0,$$

the line rises from left to right.



If

$$m < 0,$$

the line falls from left to right.



If

$$m = 0,$$

Positive and Negative Gradients (continued)

the line is horizontal.

The magnitude $|m|$ indicates how rapidly y changes as x changes.

Gradient From Two Points

Suppose a line passes through

$$(x_1, y_1)$$

and

$$(x_2, y_2).$$

Then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

provided

$$x_2 \neq x_1.$$

For example, through

Gradient From Two Points (continued)

$$(2, 3) \quad \text{and} \quad (6, 11),$$

we have

$$m = \frac{11 - 3}{6 - 2} = \frac{8}{4} = \boxed{2}.$$

The Sign Trap

Suppose the points are

$$(-2, 5) \quad \text{and} \quad (-4, -7).$$

Then

$$m = \frac{-7 - 5}{-4 - (-2)} = \frac{-12}{-2} = 6.$$

You could reverse both orders:

$$m = \frac{5 - (-7)}{-2 - (-4)} = \frac{12}{2} = 6.$$

Common Trap

The order on top and bottom must be consistent.

Do not reverse the y -coordinates without also reversing the x -coordinates.

Gradient and Angle

If a non-vertical line makes an angle θ with the positive x -axis, then

$$m = \tan \theta.$$

For example,

$$\theta = 45^\circ$$

gives

$$m = \tan 45^\circ = 1.$$

Hence

$$y = x$$

has gradient 1.

Similarly,

Gradient and Angle (continued)

$$\tan 135^\circ = -1,$$

consistent with

$$y = -x.$$

Horizontal Lines

A horizontal line has constant y .

For example,

$$y = 4.$$

Every point on this line has y -coordinate 4.

Its gradient is

$$0.$$

Why?

For any two points,

$$\Delta y = 0.$$

Therefore

$$m = \frac{0}{\Delta x} = 0.$$

Vertical Lines

A vertical line has constant x .
For example,

$$x = 3.$$

Every point on the line has x -coordinate 3.
Its gradient is not defined because

$$m = \frac{\Delta y}{\Delta x}$$

would require division by

$$\Delta x = 0.$$

Common Trap

Do not treat a vertical line as having an ordinary numerical gradient.
Handle it as a special case.

The Coordinate Axes

The x -axis consists of all points

$$(x, 0).$$

Therefore its equation is

$$y = 0.$$

The y -axis consists of all points

$$(0, y).$$

Therefore its equation is

$$x = 0.$$

These simple equations appear surprisingly often inside more complicated coordinate-geometry problems.

Point-Gradient Form

Suppose a line has gradient m and passes through

$$(x_1, y_1).$$

For any other point (x, y) on the line,

$$\frac{y - y_1}{x - x_1} = m.$$

Therefore

$$y - y_1 = m(x - x_1).$$

This is the **point-gradient form** of a straight line.

Example: Point and Gradient

Find the equation of the line with gradient 3 passing through

$$(2, 5).$$

Using

$$y - y_1 = m(x - x_1),$$

we obtain

$$y - 5 = 3(x - 2).$$

Therefore

$$y - 5 = 3x - 6,$$

so

$$y = 3x - 1.$$

Example: Point and Gradient (continued)

Check:

$$x = 2 \Rightarrow y = 6 - 1 = 5.$$

Correct.

Finding a Line From Two Points

Suppose a line passes through

$$(1, 4) \quad \text{and} \quad (5, 12).$$

First calculate the gradient:

$$m = \frac{12 - 4}{5 - 1} = 2.$$

Now use one point:

$$y - 4 = 2(x - 1).$$

Therefore

$$y = 2x + 2.$$

Hence

$$\boxed{y = 2x + 2}.$$

Alternative Direct Method

Given two points

$$(x_1, y_1), \quad (x_2, y_2),$$

we know

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This can be rearranged directly.

For example, through

$$(1, 4), \quad (5, 12),$$

write

$$\frac{y - 4}{x - 1} = \frac{12 - 4}{5 - 1} = 2.$$

Alternative Direct Method (continued)

Hence

$$y - 4 = 2(x - 1).$$

The General Form of a Line

A straight line may also be written as

$$ax + by + c = 0.$$

For example,

$$2x + 3y - 12 = 0.$$

Rearranging:

$$3y = -2x + 12,$$

so

$$y = -\frac{2}{3}x + 4.$$

Therefore

The General Form of a Line (continued)

$$m = -\frac{2}{3}, \quad y\text{-intercept} = 4.$$

A Notation Warning

Compare

$$y = mx + c$$

with

$$ax + by + c = 0.$$

Both use the letter c , but its role is not automatically the same.

In

$$y = mx + c,$$

c is the y -intercept.

But in

$$ax + by + c = 0,$$

the constant c is simply one coefficient in the equation.

A Notation Warning (continued)

Common Trap

Never assume the c in $ax + by + c = 0$ is the y -intercept.

Parallel Lines

Two distinct non-vertical lines are parallel when they have the same gradient.
If

$$y = m_1x + c_1$$

and

$$y = m_2x + c_2,$$

then

$$m_1 = m_2$$

is the gradient condition for parallelism.
For example,

$$y = 3x + 1$$

and

Parallel Lines (continued)

$$y = 3x - 7$$

are parallel.

Why Same Gradient?

Gradient measures the rate of vertical change relative to horizontal change.
If two lines have exactly the same direction, then this rate must be identical.
Thus parallel lines satisfy

$$m_1 = m_2.$$

If they also have the same intercept, they are not distinct parallel lines: they are the **same line**.

Equation of a Parallel Line

Find the line parallel to

$$y = 4x - 3$$

passing through

$$(2, 7).$$

A parallel line must have gradient

$$m = 4.$$

Therefore

$$y - 7 = 4(x - 2).$$

Hence

$$y - 7 = 4x - 8,$$

Equation of a Parallel Line (continued)

so

$$y = 4x - 1.$$

Perpendicular Lines

For two non-vertical, non-horizontal perpendicular lines,

$$m_1 m_2 = -1.$$

Equivalently,

$$m_2 = -\frac{1}{m_1}.$$

For example, if

$$m_1 = 2,$$

then a perpendicular gradient is

$$m_2 = -\frac{1}{2}.$$

Because

Perpendicular Lines (continued)

$$2\left(-\frac{1}{2}\right) = -1.$$

Negative Reciprocal

Suppose a line has gradient

$$\frac{3}{5}.$$

The perpendicular gradient is

$$-\frac{5}{3}.$$

Notice two changes:

$$\frac{3}{5} \longrightarrow \frac{5}{3}$$

and

$$+ \longrightarrow -.$$

Thus we take the **negative reciprocal**.

Negative Reciprocal (continued)

Common Trap

Do not merely change the sign.

The perpendicular gradient to 3 is not -3 .

It is

$$-\frac{1}{3}.$$

Equation of a Perpendicular Line

Find the line perpendicular to

$$y = 2x + 5$$

and passing through

$$(4, 1).$$

The original gradient is

$$2.$$

Therefore the perpendicular gradient is

$$-\frac{1}{2}.$$

Using point-gradient form,

$$y - 1 = -\frac{1}{2}(x - 4).$$

Equation of a Perpendicular Line (continued)

Hence

$$y = -\frac{1}{2}x + 3.$$

The Vertical-Horizontal Special Case

The formula

$$m_1 m_2 = -1$$

does not directly handle vertical lines because their gradients are undefined.
But geometrically:

vertical line $\perp$ horizontal line.

Thus

$$x = 4$$

is perpendicular to

$$y = 7.$$

The Vertical-Horizontal Special Case (continued)

TMUA Thinking

Always inspect whether a line is vertical or horizontal before blindly applying gradient formulae.

Intersections and Simultaneous Equations

Consider

$$y = 2x + 1$$

and

$$y = -x + 7.$$

At an intersection, the same (x, y) satisfies both equations.
Therefore

$$2x + 1 = -x + 7.$$

Hence

$$3x = 6,$$

so

Intersections and Simultaneous Equations (continued)

$$x = 2.$$

Then

$$y = 5.$$

Therefore the lines intersect at

$$(2, 5).$$

Algebraic and Geometric Meaning

Solving

$$\begin{cases} L_1(x, y) = 0, \\ L_2(x, y) = 0 \end{cases}$$

means finding points belonging to both lines.

Geometrically, these are precisely the intersection points.

Thus

$$\text{simultaneous solution} \longleftrightarrow \text{graph intersection}.$$

This idea extends far beyond straight lines.

How Many Solutions Can Two Lines Have?

Two straight lines can have:

0 intersections

if they are distinct and parallel;

1 intersection

if their gradients differ;
or

infinitely many intersections

if they are actually the same line.

This gives a geometric interpretation of systems of linear equations.

The Geometry Behind a Circle

A circle is the set of all points a fixed distance from a fixed centre.
Suppose the centre is

$(0, 0)$

and the radius is r .

A point (x, y) lies on the circle exactly when

$$\sqrt{x^2 + y^2} = r.$$

Squaring gives

$$x^2 + y^2 = r^2.$$

Thus geometry naturally produces the algebraic equation.

The Unit Circle

If

$$r = 1,$$

then

$$x^2 + y^2 = 1.$$

For example,

$$(1, 0)$$

lies on the circle because

$$1^2 + 0^2 = 1.$$

Similarly,

$$(0, -1)$$

The Unit Circle (continued)

lies on it.

But

$$(1, 1)$$

does not because

$$1^2 + 1^2 = 2 \neq 1.$$

Inside, On, or Outside?

For the circle

$$x^2 + y^2 = r^2,$$

a point satisfies

$$x^2 + y^2 < r^2$$

if it lies **inside** the circle;

$$x^2 + y^2 = r^2$$

if it lies **on** the circle;
and

$$x^2 + y^2 > r^2$$

if it lies **outside** the circle.

This is simply a comparison of the point's distance from the centre with the radius.

Circle With Centre (a, b)

Suppose a circle has centre

$$(a, b)$$

and radius

$$r.$$

The distance from (x, y) to (a, b) must equal r :

$$\sqrt{(x - a)^2 + (y - b)^2} = r.$$

Therefore

$$\boxed{(x - a)^2 + (y - b)^2 = r^2}.$$

This is the standard centre-radius form.

Reading a Circle Immediately

Consider

$$(x - 2)^2 + (y - 3)^2 = 25.$$

Comparing with

$$(x - a)^2 + (y - b)^2 = r^2,$$

we obtain

$$a = 2, \quad b = 3,$$

and

$$r^2 = 25.$$

Therefore

centre = (2, 3)

Reading a Circle Immediately (continued)

and

$r = 5$

.

The Sign Trap in Circle Centres

Consider

$$(x - 2)^2 + (y + 4)^2 = 18.$$

Rewrite mentally:

$$(y + 4)^2 = (y - (-4))^2.$$

Therefore the centre is

$$(2, -4).$$

The radius is

$$\sqrt{18} = 3\sqrt{2}.$$

Hence

$$r = 3\sqrt{2}.$$

The Sign Trap in Circle Centres (continued)

Common Trap

The signs inside the brackets are opposite to the coordinates of the centre.

Radius or Radius Squared?

For

$$(x - a)^2 + (y - b)^2 = 49,$$

we have

$$r^2 = 49.$$

Therefore

$$\boxed{r = 7},$$

not 49.

Similarly, for

$$(x - a)^2 + (y - b)^2 = 18,$$

the radius is

Radius or Radius Squared? (continued)

$$\boxed{\sqrt{18} = 3\sqrt{2}}.$$

Common Trap

The right-hand side is r^2 , not r .

Another Form of a Circle

A circle may appear as

$$x^2 + y^2 + cx + dy + e = 0.$$

The centre and radius are not immediately visible.

The main technique is:

$$\text{complete the square in } x \text{ and } y.$$

We aim to transform the equation into

$$(x - a)^2 + (y - b)^2 = r^2.$$

Completing the Square: Reminder

Recall

$$x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

For example,

$$x^2 + 4x = (x + 2)^2 - 4.$$

And

$$y^2 - 6y = (y - 3)^2 - 9.$$

These identities are exactly what we need for expanded circle equations.

Example: Find Centre and Radius

Consider

$$x^2 + y^2 + 4x + 2y - 12 = 0.$$

Group the terms:

$$(x^2 + 4x) + (y^2 + 2y) - 12 = 0.$$

Complete the squares:

$$(x + 2)^2 - 4 + (y + 1)^2 - 1 - 12 = 0.$$

Therefore

$$(x + 2)^2 + (y + 1)^2 = 17.$$

Hence

$$\boxed{\text{centre} = (-2, -1)}$$

Example: Find Centre and Radius (continued)

and

$$\boxed{r = \sqrt{17}}.$$

General Centre Formula

Suppose

$$x^2 + y^2 + cx + dy + e = 0.$$

Complete the squares:

$$\left(x + \frac{c}{2}\right)^2 - \frac{c^2}{4} + \left(y + \frac{d}{2}\right)^2 - \frac{d^2}{4} + e = 0.$$

Thus

$$\left(x + \frac{c}{2}\right)^2 + \left(y + \frac{d}{2}\right)^2 = \frac{c^2 + d^2}{4} - e.$$

Therefore the centre is

$$\left(-\frac{c}{2}, -\frac{d}{2}\right).$$

General Radius Formula

From

$$\left(x + \frac{c}{2}\right)^2 + \left(y + \frac{d}{2}\right)^2 = \frac{c^2 + d^2}{4} - e,$$

we have

$$r^2 = \frac{c^2 + d^2}{4} - e.$$

Therefore

$$r = \sqrt{\frac{c^2 + d^2}{4} - e}$$

provided the expression under the square root is positive.

But this immediately raises an important question:

General Radius Formula (continued)

What if $r^2 < 0$?

Not Every “Circle-Looking” Equation Is a Circle

Consider

$$x^2 + y^2 - 4x - 6y + 20 = 0.$$

Complete the squares:

$$(x - 2)^2 - 4 + (y - 3)^2 - 9 + 20 = 0.$$

Hence

$$(x - 2)^2 + (y - 3)^2 = -7.$$

But the left-hand side is a sum of squares, so

$$(x - 2)^2 + (y - 3)^2 \geq 0.$$

It can never equal -7 .

Therefore there are

Not Every “Circle-Looking” Equation Is a Circle (continued)

no real points satisfying the equation.

Three Possibilities

After completing the square, suppose we obtain

$$(x - a)^2 + (y - b)^2 = K.$$

Then:

$$K > 0$$

gives a genuine circle of radius $\sqrt{K}$;

$$K = 0$$

gives the single point

$$(a, b);$$

and

$$K < 0$$

Three Possibilities (continued)

gives no real points.

TMUA Thinking

Do not automatically assume that every expression of the form

$$x^2 + y^2 + cx + dy + e = 0$$

represents a real circle.

A Line Meeting a Circle

Consider the circle

$$x^2 + y^2 = 25$$

and the line

$$y = 3.$$

Substitute the line equation into the circle:

$$x^2 + 3^2 = 25.$$

Thus

$$x^2 = 16,$$

so

$$x = \pm 4.$$

A Line Meeting a Circle (continued)

Therefore the intersection points are

$$(4, 3)$$

and

$$(-4, 3).$$

How Many Intersections?

A straight line and a circle may have:

$$2$$

intersection points;

$$1$$

intersection point, when the line is tangent;
or

$$0$$

real intersection points.

Algebraically, substituting the line into the circle normally produces a quadratic equation.
The number of real roots of that quadratic tells us the number of intersections.

Discriminant Interpretation

Suppose substitution produces

$$Ax^2 + Bx + C = 0.$$

Then

$$\Delta = B^2 - 4AC.$$

Therefore:

$$\Delta > 0 \iff 2 \text{ intersections,}$$

$$\Delta = 0 \iff 1 \text{ intersection,}$$

$$\Delta < 0 \iff 0 \text{ real intersections.}$$

This connects algebra directly to geometry.

TMUA Challenge 1

Find the equation of the line through

$$(-2, 5)$$

and

$$(-4, -7).$$

Gradient:

$$m = \frac{-7 - 5}{-4 - 2} = \frac{-12}{-2} = 6.$$

Using $(-2, 5)$,

$$y - 5 = 6(x + 2).$$

Thus

$$y = 6x + 17.$$

TMUA Challenge 1 (continued)

Therefore

$$y = 6x + 17.$$

TMUA Challenge 2

Find the equation of the line through

$$(3, -7)$$

and

$$(8, -7).$$

The y -coordinates are identical.
Therefore the line is horizontal.
Hence immediately,

$$y = -7.$$

TMUA Thinking

Before calculating a gradient mechanically, inspect the coordinates.
Sometimes the geometry gives the answer immediately.

TMUA Challenge 3

A line

$$L : 3x - 2y + 7 = 0$$

is given.

Find its gradient.

Rearrange:

$$-2y = -3x - 7.$$

Hence

$$y = \frac{3}{2}x + \frac{7}{2}.$$

Therefore

$$m = \frac{3}{2}.$$

TMUA Challenge 3 (continued)

A perpendicular line therefore has gradient

$$-\frac{2}{3}.$$

TMUA Challenge 4

Find the line perpendicular to

$$3x - 2y + 7 = 0$$

and passing through

$$(3, 4).$$

The original gradient is

$$\frac{3}{2}.$$

Hence the perpendicular gradient is

$$-\frac{2}{3}.$$

Using point-gradient form,

TMUA Challenge 4 (continued)

$$y - 4 = -\frac{2}{3}(x - 3).$$

Therefore

$$y = -\frac{2}{3}x + 6.$$

Equivalently,

$$\boxed{2x + 3y - 18 = 0}.$$

TMUA Challenge 5

Find the centre and radius of

$$x^2 + y^2 - 8x + 6y - 11 = 0.$$

Complete the square:

$$(x^2 - 8x) + (y^2 + 6y) = 11.$$

Thus

$$(x - 4)^2 - 16 + (y + 3)^2 - 9 = 11.$$

Hence

$$(x - 4)^2 + (y + 3)^2 = 36.$$

Therefore

$\text{centre} = (4, -3)$

TMUA Challenge 5 (continued)

and

$r = 6$

.

TMUA Challenge 6

Determine whether

$$x^2 + y^2 + 2x - 4y + 10 = 0$$

represents a real circle.

Complete the square:

$$(x + 1)^2 - 1 + (y - 2)^2 - 4 + 10 = 0.$$

Therefore

$$(x + 1)^2 + (y - 2)^2 = -5.$$

The left-hand side cannot be negative.

Hence

there is no real circle.

Avoid Unnecessary Algebra

Suppose a line passes through

$$(2, 5)$$

and is parallel to

$$y = 7x - 4.$$

You already know its gradient:

$$m = 7.$$

So immediately,

$$y - 5 = 7(x - 2).$$

There is no need to introduce multiple unknown coefficients.

Avoid Unnecessary Algebra (continued)

TMUA Thinking

Before solving, identify which geometric information already determines part of the equation.

A Line Is Determined by Two Pieces of Information

Examples of sufficient information include:

one point + gradient

or

two distinct points.

Similarly, knowing

one point + parallel to a known line

gives the gradient.

And

one point + perpendicular to a known line

also gives the gradient.

The problem is then reduced to point-gradient form.

Think Geometrically Before Manipulating

Suppose

$$L_1 : y = 4x + 3$$

and

$$L_2 : 8x - 2y + 5 = 0.$$

Before solving for an intersection, rewrite L_2 :

$$2y = 8x + 5,$$

so

$$y = 4x + \frac{5}{2}.$$

Both lines have gradient 4, but different intercepts.

Therefore

Think Geometrically Before Manipulating (continued)

the lines are parallel

and hence

there is no intersection.

No simultaneous-equation calculation is necessary.

Finding a Circle From Its Centre and a Point

A circle has centre

$$(2, -1)$$

and passes through

$$(5, 3).$$

First find r^2 :

$$r^2 = (5 - 2)^2 + [3 - (-1)]^2.$$

Thus

$$r^2 = 3^2 + 4^2 = 25.$$

Therefore the equation is

$$(x - 2)^2 + (y + 1)^2 = 25.$$

Does a Point Lie on a Circle?

Consider

$$(x - 1)^2 + (y + 2)^2 = 20.$$

Does

$$(3, 2)$$

lie on the circle?

Substitute:

$$(3 - 1)^2 + (2 + 2)^2 = 4 + 16 = 20.$$

Yes.

Therefore

$$(3, 2) \text{ lies on the circle.}$$

This simple substitution principle is fundamental.

Inside or Outside?

For

$$(x - 1)^2 + (y + 2)^2 = 20,$$

consider the point

$$(1, 0).$$

Compute

$$(1 - 1)^2 + (0 + 2)^2 = 4.$$

Since

$$4 < 20,$$

the point lies

inside the circle.

We do not need to sketch the entire circle.

The Straight-Line Decision Process

When you see a line problem, ask:

- ① Is the line horizontal or vertical?
- ② Is the gradient already given?
- ③ Can I calculate the gradient from two points?
- ④ Is another line parallel?
- ⑤ Is another line perpendicular?
- ⑥ Which point lies on the required line?
- ⑦ Which equation form is most convenient?

Usually the fastest route is

$$\text{find } m \longrightarrow y - y_1 = m(x - x_1).$$

The Circle Decision Process

When you see a circle equation, ask:

- ① Is it already in centre-radius form?
- ② Can I read the centre immediately?
- ③ Is the right-hand side r or r^2 ?
- ④ Do I need to complete the square?
- ⑤ Does the resulting r^2 make sense?
- ⑥ Is a given point inside, on, or outside?
- ⑦ Is there also a line to intersect with the circle?

The target form is

$$(x - a)^2 + (y - b)^2 = r^2.$$

Coordinate Geometry Is Translation

Many TMUA coordinate problems are really about translating between different mathematical languages:

equation $\leftrightarrow$ graph

gradient $\leftrightarrow$ direction

simultaneous equations $\leftrightarrow$ intersections

distance $\leftrightarrow$ circle equation

discriminant $\leftrightarrow$ number of intersections

Part 4 Formula Map

For two points,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form:

$$y - y_1 = m(x - x_1)$$

Parallel lines:

$$m_1 = m_2$$

Perpendicular lines:

$$m_1 m_2 = -1$$

Circle:

Part 4 Formula Map (continued)

$$(x - a)^2 + (y - b)^2 = r^2$$

Part 4: What You Should Now Know

You should now be comfortable with:

- interpreting gradient;
- finding gradient from two points;
- equations of straight lines;
- point-gradient form;
- $y = mx + c$;
- $ax + by + c = 0$;
- horizontal and vertical lines;
- parallel lines;
- perpendicular lines;
- intersections of lines;
- geometric meaning of simultaneous equations;
- equations of circles;
- centre and radius;
- completing the square;
- recognising impossible real-circle equations.

Part 4 Self-Check

Can You Explain These Without Looking Back?

- 1 What does gradient really measure?
- 2 Why is a vertical line's gradient undefined?
- 3 How do you find a line from two points?
- 4 Why do parallel lines have equal gradients?
- 5 Why do perpendicular gradients multiply to -1 ?
- 6 What does c mean in $y = mx + c$?
- 7 How do you read a circle's centre from standard form?
- 8 Why is the radius the square root of the RHS?
- 9 How does completing the square reveal a circle?
- 10 What does a negative value of r^2 mean?

TMUA Habits for Coordinate Geometry

- ① Sketch mentally before calculating.
- ② Look for horizontal and vertical special cases.
- ③ Keep coordinate subtraction order consistent.
- ④ Use point-gradient form efficiently.
- ⑤ Read parallel/perpendicular information as gradient information.
- ⑥ Translate intersections into simultaneous equations.
- ⑦ Complete squares carefully.
- ⑧ Check whether r^2 is actually non-negative.
- ⑨ Interpret algebraic answers geometrically.

TMUA Thinking

The strongest solutions move freely between the diagram and the equation.

Coming in Part 5

TMUA Mathematics Masterclass

Part 5: Trigonometry

We will study:

- sine, cosine and tangent;
- exact values;
- radians;
- trigonometric graphs;
- identities;
- solving trigonometric equations;
- sine rule;
- cosine rule;
- area of a triangle;

- transformations;
- multiple solutions;
- TMUA-style trigonometric reasoning.

See the Geometry in the Algebra.

And see the algebra hidden inside the geometry.

Coordinates → Equations → Geometry → Reasoning

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 5: Trigonometry

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Trigonometry Is Much More Than SOH-CAH-TOA

At an introductory level, trigonometry begins with ratios in right-angled triangles. But for TMUA, we need a broader viewpoint. Trigonometry connects

angles

with

lengths,

circles,

graphs,

functions,

equations.

Key Idea

The goal is not simply to remember trigonometric formulae.

The goal is to understand how trigonometric relationships encode geometry.

Part 5 Roadmap

We will study:

- ➊ Right-angled triangle trigonometry
- ➋ Exact trigonometric values
- ➌ Area of a triangle
- ➍ The sine rule
- ➎ The ambiguous sine-rule case
- ➏ The cosine rule
- ➐ Choosing between sine and cosine rules
- ➑ Radian measure
- ➒ Arc length
- ➓ Sector and segment area
- ➔ Sine, cosine and tangent graphs
- ➕ Symmetry and periodicity
- ➖ Graph transformations

Part 5 Roadmap (continued)

- ➗ Trigonometric signs
- ➙ Fundamental identities
- ➚ Solving trigonometric equations
- ➛ TMUA-style structural reasoning

The Three Basic Ratios

For an acute angle θ in a right-angled triangle,

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}.$$

These are often remembered through

SOH-CAH-TOA.

But understanding the ratios is more important than remembering the mnemonic.

The Labels Depend on the Angle

The hypotenuse is fixed:

the side opposite the right angle.

But “opposite” and “adjacent” depend on which acute angle we are considering.

If we change the reference angle,

opposite $\longleftrightarrow$ adjacent.

Common Trap

Always identify the reference angle before deciding which side is opposite and which is adjacent.

A Useful Relationship

From the right-triangle definitions,

$$\sin \theta = \frac{O}{H}, \quad \cos \theta = \frac{A}{H}.$$

Therefore

$$\frac{\sin \theta}{\cos \theta} = \frac{O/H}{A/H} = \frac{O}{A}.$$

But

$$\frac{O}{A} = \tan \theta.$$

Hence

$$\boxed{\tan \theta = \frac{\sin \theta}{\cos \theta}}.$$

This identity will become important later.

Exact Values Matter

You should know, or be able to derive quickly, the values of

$$\sin \theta, \quad \cos \theta, \quad \tan \theta$$

for the standard angles

$$0^\circ, \quad 30^\circ, \quad 45^\circ, \quad 60^\circ, \quad 90^\circ.$$

In radians these are

$$0, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}.$$

TMUA Thinking

Exact values should become part of your mathematical vocabulary, not something you need to rediscover slowly in every question.

The 45° Triangle

Consider an isosceles right-angled triangle with shorter sides

$$1, \quad 1.$$

By Pythagoras,

$$\text{hypotenuse} = \sqrt{2}.$$

Therefore

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and similarly,

$$\cos 45^\circ = \frac{\sqrt{2}}{2}.$$

Also,

The 45° Triangle (continued)

$$\tan 45^\circ = 1.$$

The 30°–60°–90° Triangle

Start with an equilateral triangle of side 2.

Bisect it.

The resulting right triangle has sides

$$1, \quad \sqrt{3}, \quad 2.$$

Hence

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

And

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3}.$$

Standard Exact Values

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Common Trap

$\tan 90^\circ$ is undefined because

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

and $\cos 90^\circ = 0$.

The Familiar Area Formula

For a triangle,

$$A = \frac{1}{2}bh.$$

Here b is the base and h is the **perpendicular height**.
The height is not necessarily one of the sides of the triangle.

This elementary formula is the starting point for one of the most useful trigonometric area formulae.

Deriving the Trigonometric Area Formula

Suppose two sides have lengths a and b , with included angle C .
Taking a as the base, the perpendicular height is

$$h = b \sin C.$$

Therefore

$$A = \frac{1}{2}ah = \frac{1}{2}a(b \sin C).$$

Hence

$$A = \frac{1}{2}ab \sin C.$$

Equivalent Area Formulae

By choosing different pairs of sides,

$$A = \frac{1}{2}ab \sin C$$

$$A = \frac{1}{2}bc \sin A$$

and

$$A = \frac{1}{2}ca \sin B.$$

Notice the structure:

two sides + sine of their included angle.

Example: Area From Two Sides and an Angle

Suppose

$$a = 8, \quad b = 5, \quad C = 30^\circ.$$

Then

$$A = \frac{1}{2}(8)(5) \sin 30^\circ.$$

Since

$$\sin 30^\circ = \frac{1}{2},$$

we obtain

$$A = 20 \cdot \frac{1}{2} = \boxed{10}.$$

Where Does the Sine Rule Come From?

The same triangle has the same area regardless of which side we choose as its base.

Thus

$$\frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B.$$

Multiply by 2:

$$ab \sin C = bc \sin A = ca \sin B.$$

Divide by abc :

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This is the sine rule.

The More Familiar Form

The sine rule is often written

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Remember the pairing:

$$a \leftrightarrow A, \quad b \leftrightarrow B, \quad c \leftrightarrow C.$$

Each side must be paired with its **opposite angle**.

When Is the Sine Rule Useful?

The sine rule is particularly useful when we know

two angles and one side

and want another side.

It is also useful when we know

two sides and a non-included angle

and want another angle.

The second situation requires special care because it may lead to the **ambiguous case**.

Example: Finding a Side

Suppose

$$A = 30^\circ, \quad B = 45^\circ, \quad a = 6.$$

Find b .

Using

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

we have

$$\frac{6}{\sin 30^\circ} = \frac{b}{\sin 45^\circ}.$$

Thus

$$b = 6 \frac{\sin 45^\circ}{\sin 30^\circ}.$$

Therefore

Example: Finding a Side (continued)

$$b = 6 \frac{\sqrt{2}/2}{1/2} = \boxed{6\sqrt{2}}.$$

Why Can the Sine Rule Be Ambiguous?

Suppose we obtain

$$\sin B = k.$$

For many positive values of k ,

$$\sin B = \sin(180^\circ - B).$$

For example,

$$\sin 30^\circ = \sin 150^\circ = \frac{1}{2}.$$

Therefore knowing the sine of an angle does not always uniquely determine the angle between 0° and 180° .

Source-Style Example

Suppose

$$A = 30^\circ, \quad a = 4, \quad b = 6.$$

Using the sine rule,

$$\frac{\sin B}{6} = \frac{\sin 30^\circ}{4}.$$

Hence

$$\sin B = \frac{6}{4} \cdot \frac{1}{2} = \frac{3}{4}.$$

One solution is approximately

$$B \approx 48.6^\circ.$$

But another is

$$B \approx 180^\circ - 48.6^\circ = 131.4^\circ.$$

Do Both Angles Work?

We must also satisfy

$$A + B + C = 180^\circ.$$

Here

$$A = 30^\circ.$$

For

$$B \approx 48.6^\circ,$$

we have

$$A + B < 180^\circ.$$

For

$$B \approx 131.4^\circ,$$

Do Both Angles Work? (continued)

we also have

$$30^\circ + 131.4^\circ < 180^\circ.$$

Therefore both triangles are geometrically possible.

Key Idea

An inverse sine calculation gives a candidate angle.
Geometry determines whether the alternative angle is also valid.

The Ambiguous Case: Possible Outcomes

Depending on the side lengths and given angle, the data may describe

0 triangles,

1 triangle,

or

2 triangles.

This is why blindly accepting the calculator's inverse-sine answer is dangerous.

TMUA Thinking

Whenever using the sine rule to find an angle, ask:

Could the supplementary angle also work?

The Cosine Rule

For a triangle with sides a, b, c , where A is opposite a ,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly,

$$b^2 = c^2 + a^2 - 2ca \cos B,$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

The structure matters more than memorising three separate formulae.

Cosine Rule as Corrected Pythagoras

Recall Pythagoras:

$$a^2 = b^2 + c^2$$

for a right-angled triangle.

The cosine rule says

$$a^2 = b^2 + c^2 - \underbrace{2bc \cos A}_{\text{correction term}}.$$

Thus we can think of it as

$$\text{Pythagoras} + \text{an angle-dependent correction}.$$

Why Pythagoras Is a Special Case

Set

$$A = 90^\circ.$$

Then

$$\cos 90^\circ = 0.$$

Therefore

$$a^2 = b^2 + c^2 - 2bc(0),$$

so

$$a^2 = b^2 + c^2.$$

Thus Pythagoras' theorem is contained inside the cosine rule.

When Should We Use the Cosine Rule?

The cosine rule is particularly useful when we know

three sides

and want an angle,
or when we know

two sides and the included angle

and want the third side.

For a right-angled triangle, ordinary right-triangle methods are often simpler.

Example: Finding the Third Side

Suppose

$$b = 5, \quad c = 7, \quad A = 60^\circ.$$

Then

$$a^2 = 5^2 + 7^2 - 2(5)(7) \cos 60^\circ.$$

Hence

$$a^2 = 25 + 49 - 70 \left(\frac{1}{2} \right) = 39.$$

Therefore

$$a = \sqrt{39}.$$

Finding an Angle With the Cosine Rule

Starting from

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

rearrange:

$$2bc \cos A = b^2 + c^2 - a^2.$$

Hence

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

It is usually better to understand how to rearrange the main cosine-rule equation than to treat this as an unrelated formula.

Which Formula Should I Use?

Information	Likely Method
Right triangle	sin, cos, tan or Pythagoras
Two sides + included angle	Cosine rule
Three sides	Cosine rule
Two angles + one side	Sine rule
Two sides + non-included angle	Sine rule
Two sides + included angle, area	$\frac{1}{2}ab \sin C$

TMUA Thinking

Do not begin by asking, "Which formula do I remember?"
Ask, "What information do I have, and what do I need?"

Why Do We Need Another Angle Measure?

Degrees divide one revolution into

$$360^\circ.$$

This convention is useful, but the number 360 is not intrinsic to the geometry of a circle.
Radians define angle directly through

arc length and radius.

This makes radians especially natural in higher mathematics.

What Is One Radian?

Imagine a circle of radius r .

An angle of 1 radian subtends an arc whose length is also r .

Thus

$$\theta = 1 \text{ rad} \iff s = r.$$

More generally,

$$\theta = \frac{s}{r}.$$

Hence

$$s = r\theta$$

when θ is measured in radians.

One Full Revolution

The circumference of a circle is

$$2\pi r.$$

Using

$$s = r\theta,$$

a complete revolution satisfies

$$2\pi r = r\theta.$$

Therefore

$$\theta = 2\pi.$$

Hence

$$360^\circ = 2\pi \text{ radians}.$$

One Full Revolution (continued)

Dividing by 2,

$$180^\circ = \pi \text{ radians}.$$

Degrees to Radians

Using

$$180^\circ = \pi,$$

we obtain

$$\theta^\circ = \theta \frac{\pi}{180} \text{ radians}.$$

For example,

$$60^\circ = 60 \frac{\pi}{180} = \frac{\pi}{3}.$$

Similarly,

$$45^\circ = \frac{\pi}{4}.$$

Radians to Degrees

Conversely,

$$\theta \text{ radians} = \theta \frac{180^\circ}{\pi}.$$

For example,

$$\frac{2\pi}{3}$$

corresponds to

$$\frac{2\pi}{3} \frac{180^\circ}{\pi} = \boxed{120^\circ}.$$

Standard Conversions

Degrees	Radians
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

These should be recognised immediately.

Arc Length

If a sector has radius r and angle θ radians, then

$$s = r\theta.$$

For example, if

$$r = 6$$

and

$$\theta = \frac{\pi}{3},$$

then

$$s = 6 \cdot \frac{\pi}{3} = 2\pi.$$

Why $s = r\theta$?

A complete circle corresponds to

$$2\pi$$

radians and has circumference

$$2\pi r.$$

An angle θ therefore represents the fraction

$$\frac{\theta}{2\pi}$$

of a complete revolution.

Hence

$$s = \frac{\theta}{2\pi}(2\pi r) = r\theta.$$

Area of a Sector

The area of the complete circle is

$$\pi r^2.$$

A sector of angle θ radians represents the fraction

$$\frac{\theta}{2\pi}$$

of the circle.

Therefore

$$A = \frac{\theta}{2\pi} \pi r^2.$$

Hence

$$A = \frac{1}{2} r^2 \theta.$$

Example: Sector Area

Suppose

$$r = 4$$

and

$$\theta = \frac{3\pi}{4}.$$

Then

$$A = \frac{1}{2} (4^2) \frac{3\pi}{4}.$$

Thus

$$A = 8 \cdot \frac{3\pi}{4} = \boxed{6\pi}.$$

Area of a Circular Segment

A circular segment can often be viewed as

$$\boxed{\text{sector} - \text{triangle}}.$$

For radius r and central angle θ ,

$$A_{\text{sector}} = \frac{1}{2}r^2\theta.$$

The triangle formed by the two radii has area

$$A_{\triangle} = \frac{1}{2}r^2 \sin \theta.$$

Therefore, for the corresponding minor segment,

$$\boxed{A_{\text{segment}} = \frac{1}{2}r^2(\theta - \sin \theta)}.$$

From Triangles to Functions

We now move from thinking about

$$\sin \theta, \quad \cos \theta, \quad \tan \theta$$

as ratios in a triangle to thinking about

$$y = \sin x, \quad y = \cos x, \quad y = \tan x$$

as functions.

This shift is essential.

Key Idea

A trigonometric function accepts an angle as its input and produces a number as its output.

The Sine Function

For

$$y = \sin x,$$

the values oscillate between

$$-1$$

and

$$1.$$

Thus

$$\boxed{-1 \leq \sin x \leq 1}.$$

Important points include

$$\sin 0 = 0,$$

The Sine Function (continued)

$$\sin \frac{\pi}{2} = 1,$$

$$\sin \pi = 0,$$

$$\sin \frac{3\pi}{2} = -1,$$

and

$$\sin 2\pi = 0.$$

Periodicity of Sine

The sine graph repeats every

$$2\pi.$$

Therefore

$$\sin(x + 2\pi) = \sin x.$$

More generally,

$$\sin(x + 2k\pi) = \sin x, \quad k \in \mathbb{Z}.$$

The period is therefore

$$2\pi.$$

Symmetry of Sine

The sine function satisfies

$$\sin(-x) = -\sin x.$$

Thus sine is an **odd function**.

Geometrically, its graph has rotational symmetry of 180° about the origin.

The Cosine Function

For

$$y = \cos x,$$

again

$$\boxed{-1 \leq \cos x \leq 1}.$$

Important points are

$$\cos 0 = 1,$$

$$\cos \frac{\pi}{2} = 0,$$

$$\cos \pi = -1,$$

$$\cos \frac{3\pi}{2} = 0,$$

The Cosine Function (continued)

and

$$\cos 2\pi = 1.$$

Periodicity and Symmetry of Cosine

Cosine repeats every 2π :

$$\cos(x + 2\pi) = \cos x.$$

It also satisfies

$$\cos(-x) = \cos x.$$

Therefore cosine is an **even function**.
Its graph is symmetric about the y -axis.

The Tangent Function

Recall

$$\tan x = \frac{\sin x}{\cos x}.$$

Therefore tangent is undefined whenever

$$\cos x = 0.$$

This occurs at

$$x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

These values correspond to vertical asymptotes of the tangent graph.

Periodicity of Tangent

Unlike sine and cosine,

$$\tan x$$

has period

$$\pi.$$

Thus

$$\tan(x + \pi) = \tan x.$$

More generally,

$$\tan(x + k\pi) = \tan x.$$

This shorter period is extremely important when solving trigonometric equations.

Symmetry of Tangent

Tangent satisfies

$$\tan(-x) = -\tan x.$$

Therefore tangent is an odd function.
Its graph has rotational symmetry about the origin.

Compare:

$$\sin(-x) = -\sin x,$$

$$\cos(-x) = \cos x,$$

$$\tan(-x) = -\tan x.$$

Sine and Cosine as Projections

A useful geometric interpretation is to regard sine and cosine as projections.

For a directed length r making angle θ ,

$$x = r \cos \theta$$

is its horizontal component, while

$$y = r \sin \theta$$

is its vertical component.

Thus the signs of sine and cosine tell us the direction of the horizontal and vertical projections.

Signs in the Four Quadrants

For an angle measured from the positive x-axis:

Quadrant	$\sin \theta$	$\cos \theta$	$\tan \theta$
<i>I</i>	+	+	+
<i>II</i>	+	−	−
<i>III</i>	−	−	+
<i>IV</i>	−	+	−

This sign structure is often remembered through a CAST-type diagram.

TMUA Thinking

Understanding why the signs occur is more reliable than memorising a diagram without geometric meaning.

Vertical Scaling

Compare

$$y = \sin x$$

with

$$y = 2 \sin x.$$

The first has range

$$[-1, 1].$$

The second has range

$$[-2, 2].$$

Thus the coefficient outside the sine function changes the vertical scale.
For

Vertical Scaling (continued)

$$y = A \sin x,$$

the amplitude is

$$|A|.$$

Horizontal Scaling

Compare

$$y = \sin x$$

and

$$y = \sin 2x.$$

The graph of $\sin 2x$ completes its cycle twice as quickly.
Its period is

$$\boxed{\pi}.$$

More generally,

$$y = \sin(kx)$$

has period

Horizontal Scaling (continued)

$$\boxed{\frac{2\pi}{|k|}}.$$

Similarly,

$$y = \cos(kx)$$

has period

$$\boxed{\frac{2\pi}{|k|}}.$$

Tangent Under Horizontal Scaling

Since the ordinary tangent function has period π ,

$$y = \tan(kx)$$

has period

$$\frac{\pi}{|k|}.$$

For example,

$$y = \tan 2x$$

has period

$$\frac{\pi}{2}.$$

Always remember that tangent has a different fundamental period from sine and cosine.

Horizontal Translation

Consider

$$y = \sin(x - a).$$

This shifts the sine graph to the right by a .

By contrast,

$$y = \sin(x + a)$$

shifts it to the left by a .

Common Trap

Horizontal transformations often appear to work with the opposite sign from what students first expect.

Be Careful With Brackets

Compare

$$\sin(2x + \alpha)$$

with

$$\sin(2(x + \alpha)).$$

The second expression equals

$$\sin(2x + 2\alpha),$$

not

$$\sin(2x + \alpha).$$

Thus

$$\sin(2x + \alpha) \neq \sin(2(x + \alpha))$$

Be Careful With Brackets (continued)

in general.

TMUA Thinking

Read the structure of the argument before interpreting the transformation.

The Pythagorean Identity

One of the central identities is

$$\sin^2 \theta + \cos^2 \theta = 1.$$

This is closely connected to Pythagoras' theorem.

If

$$\sin \theta = \frac{O}{H}, \quad \cos \theta = \frac{A}{H},$$

then

$$\sin^2 \theta + \cos^2 \theta = \frac{O^2 + A^2}{H^2}.$$

By Pythagoras,

$$O^2 + A^2 = H^2.$$

Therefore

The Pythagorean Identity (continued)

$$\sin^2 \theta + \cos^2 \theta = 1.$$

The Identity Holds Beyond Acute Angles

The triangle derivation may initially use an acute angle.
But

$$\sin^2 \theta + \cos^2 \theta = 1$$

holds for all real θ .

One way to understand this is through the unit circle:

$$x = \cos \theta, \quad y = \sin \theta.$$

Since every point on the unit circle satisfies

$$x^2 + y^2 = 1,$$

we obtain

$$\cos^2 \theta + \sin^2 \theta = 1.$$

The Quotient Identity

The other central identity required here is

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$

Together,

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

and

$$\sin^2 \theta + \cos^2 \theta = 1$$

allow many trigonometric equations to be transformed into a simpler form.

Using the Pythagorean Identity

From

$$\sin^2 x + \cos^2 x = 1,$$

we may write

$$\cos^2 x = 1 - \sin^2 x$$

or

$$\sin^2 x = 1 - \cos^2 x.$$

This allows an equation containing both sine and cosine to sometimes be rewritten entirely in terms of one function.

Why Trigonometric Equations Are Different

Consider

$$\sin x = \frac{1}{2}.$$

An algebraic instinct might suggest finding one value of x .

But sine is periodic.

Therefore there are infinitely many real solutions.

When a question specifies an interval, our job is to identify

every solution in that interval.

Common Trap

Finding one calculator answer is not the same as solving a trigonometric equation.

Solving $\sin x = \frac{1}{2}$

Suppose

$$0^\circ < x < 360^\circ.$$

The reference angle is

$$30^\circ.$$

Sine is positive in Quadrants I and II.
Therefore

$$x = 30^\circ$$

or

$$x = 150^\circ.$$

Hence

Solving $\sin x = \frac{1}{2}$ (continued)

$$x = 30^\circ, 150^\circ.$$

Solving $\cos x = -\frac{1}{2}$

For

$$0^\circ \leq x < 360^\circ,$$

the reference angle is

$$60^\circ.$$

Cosine is negative in Quadrants II and III.

Thus

$$x = 120^\circ$$

or

$$x = 240^\circ.$$

Therefore

Solving $\cos x = -\frac{1}{2}$ (continued)

$$x = 120^\circ, 240^\circ.$$

Solving a Tangent Equation

Solve

$$\tan x = -\frac{1}{\sqrt{3}}$$

for

$$-\pi < x < \pi.$$

The reference angle is

$$\frac{\pi}{6},$$

since

$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

Tangent is negative when sine and cosine have opposite signs.
Within the given interval,

Solving a Tangent Equation (continued)

$$x = -\frac{\pi}{6}, \quad x = \frac{5\pi}{6}.$$

Always Respect the Interval

Suppose

$$\sin x = \frac{1}{2}$$

but the interval is

$$-2\pi < x < 2\pi.$$

We cannot stop after finding

$$\frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

Periodicity produces additional solutions inside the interval.

TMUA Thinking

After finding the basic solutions, systematically use periodicity and check the interval endpoints.

Example: $\sin(2x) = \frac{1}{2}$

Suppose

$$0 < x < \pi.$$

Set

$$u = 2x.$$

Then

$$0 < u < 2\pi.$$

Solve

$$\sin u = \frac{1}{2}.$$

Thus

$$u = \frac{\pi}{6}$$

Example: $\sin(2x) = \frac{1}{2}$ (continued)

or

$$u = \frac{5\pi}{6}.$$

Therefore

$$x = \frac{\pi}{12}$$

or

$$x = \frac{5\pi}{12}.$$

Transform the Interval Too

If

$$u = 2x + \frac{\pi}{3}$$

and

$$-2\pi < x < 2\pi,$$

then do not solve for u without considering its range.

From

$$-2\pi < x < 2\pi,$$

multiply by 2:

$$-4\pi < 2x < 4\pi.$$

Then add $\pi/3$:

Transform the Interval Too (continued)

$$-\frac{11\pi}{3} < u < \frac{13\pi}{3}.$$

Key Idea

When the angle expression changes, the interval changes with it.

A More Interesting Equation

Consider

$$12 \cos^2 x + 6 \sin x - 10 = 2.$$

The equation contains both

$$\cos^2 x$$

and

$$\sin x.$$

Using

$$\cos^2 x = 1 - \sin^2 x,$$

we can convert everything into a single trigonometric function.

Convert to One Function

Starting from

$$12 \cos^2 x + 6 \sin x - 10 = 2,$$

substitute

$$\cos^2 x = 1 - \sin^2 x.$$

Then

$$12(1 - \sin^2 x) + 6 \sin x - 10 = 2.$$

Simplifying,

$$12 - 12 \sin^2 x + 6 \sin x - 10 = 2.$$

Thus

$$-12 \sin^2 x + 6 \sin x = 0.$$

Convert to One Function (continued)

Equivalently,

$$12 \sin^2 x - 6 \sin x = 0.$$

Treat It Like a Quadratic

Factorise:

$$12 \sin^2 x - 6 \sin x = 0.$$

Thus

$$6 \sin x(2 \sin x - 1) = 0.$$

Hence

$$\sin x = 0$$

or

$$\sin x = \frac{1}{2}.$$

Now the original problem has become two standard trigonometric equations.

Complete the Solution

For

$$0^\circ < x < 360^\circ,$$

$$\sin x = 0$$

gives

$$x = 180^\circ.$$

And

$$\sin x = \frac{1}{2}$$

gives

$$x = 30^\circ, \quad 150^\circ.$$

Complete the Solution (continued)

Therefore

$$x = 30^\circ, 150^\circ, 180^\circ.$$

TMUA Thinking

A complicated trigonometric equation often becomes ordinary algebra after choosing the right identity.

Equations as Graph Intersections

Consider

$$\sin x = \cos x.$$

This can be interpreted as the intersection of

$$y = \sin x$$

and

$$y = \cos x.$$

Alternatively, where $\cos x \neq 0$,

$$\frac{\sin x}{\cos x} = 1,$$

so

$$\tan x = 1.$$

Equations as Graph Intersections (continued)

Therefore

$$x = \frac{\pi}{4} + k\pi.$$

This illustrates how graphical and algebraic reasoning reinforce each other.

What About $\cos x = -\sin x$?

We have

$$\cos x = -\sin x.$$

Provided $\cos x \neq 0$,

$$1 = -\tan x.$$

Thus

$$\tan x = -1.$$

Therefore

$$x = -\frac{\pi}{4} + k\pi$$

or equivalently

What About $\cos x = -\sin x$? (continued)

$$x = \frac{3\pi}{4} + k\pi.$$

Again, periodicity gives the complete family.

TMUA Challenge 1

Suppose

$$\sin \theta = \frac{3}{5}$$

and θ is acute.

Find $\cos \theta$.

Using

$$\sin^2 \theta + \cos^2 \theta = 1,$$

we have

$$\frac{9}{25} + \cos^2 \theta = 1.$$

Therefore

$$\cos^2 \theta = \frac{16}{25}.$$

TMUA Challenge 1 (continued)

Since θ is acute,

$$\cos \theta > 0.$$

Hence

$$\cos \theta = \frac{4}{5}.$$

TMUA Challenge 2

Suppose

$$\sin \theta = \frac{3}{5}$$

but now

$$\frac{\pi}{2} < \theta < \pi.$$

Again,

$$\cos^2 \theta = \frac{16}{25}.$$

But θ lies in Quadrant II, where cosine is negative.

Therefore

$$\cos \theta = -\frac{4}{5}.$$

TMUA Challenge 2 (continued)

Key Idea

The identity determines the magnitude.
The geometry determines the sign.

TMUA Challenge 3

If

$$\tan \theta = 2$$

and

$$\pi < \theta < \frac{3\pi}{2},$$

what can we say about the signs of sine and cosine?

This interval lies in Quadrant III.

Therefore

$$\sin \theta < 0, \quad \cos \theta < 0.$$

Their ratio is positive:

$$\frac{\sin \theta}{\cos \theta} > 0.$$

TMUA Challenge 3 (continued)

Hence

$$\tan \theta > 0,$$

consistent with the information given.

TMUA Challenge 4

A sector has radius 6 and arc length 5.
Find its angle in radians.
Using

$$s = r\theta,$$

we have

$$5 = 6\theta.$$

Therefore

$$\theta = \frac{5}{6}.$$

Notice that no conversion to degrees is needed.

TMUA Challenge 5

A sector has radius 4 and angle

$$\frac{2\pi}{3}.$$

Its area is

$$A = \frac{1}{2}r^2\theta.$$

Therefore

$$A = \frac{1}{2}(16)\frac{2\pi}{3} = \boxed{\frac{16\pi}{3}}.$$

Its arc length is

$$s = 4\frac{2\pi}{3} = \boxed{\frac{8\pi}{3}}.$$

TMUA Challenge 6

A triangle has sides 5 and 8, with included angle 60° .

Find its area.

Use

$$A = \frac{1}{2}ab \sin C.$$

Thus

$$A = \frac{1}{2}(5)(8) \sin 60^\circ.$$

Therefore

$$A = 20\frac{\sqrt{3}}{2} = \boxed{10\sqrt{3}}.$$

TMUA Challenge 7

The same triangle has sides 5 and 8 with included angle 60° .

Find the third side c .

Use the cosine rule:

$$c^2 = 5^2 + 8^2 - 2(5)(8) \cos 60^\circ.$$

Thus

$$c^2 = 25 + 64 - 40 = 49.$$

Hence

$$\boxed{c = 7}.$$

Trigonometry in Three Dimensions

TMUA trigonometric problems need not be confined to a single two-dimensional triangle.

A three-dimensional problem may require you to identify a useful triangle hidden inside a solid.

The usual strategy is:

$\boxed{\text{3D diagram} \rightarrow \text{relevant 2D triangle} \rightarrow \text{trigonometry}}.$

TMUA Thinking

Do not attempt to solve the whole three-dimensional figure at once.

Extract the triangle containing the required angle or length.

A Typical 3D Strategy

Suppose a cuboid has side lengths

$$a, \quad b, \quad c.$$

A face diagonal may first be found using

$$d^2 = a^2 + b^2.$$

Then the space diagonal D satisfies

$$D^2 = d^2 + c^2.$$

Hence

$$D^2 = a^2 + b^2 + c^2.$$

Once the required triangle is identified, an angle may then be found using sine, cosine or tangent.

Common Error 1: Wrong Side-Angle Pair

In the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

the side a must be opposite A .
Do not accidentally write

$$\frac{a}{\sin B}.$$

Common Trap

Before substituting numbers, mark the opposite side-angle pairs on the diagram.

Common Error 2: Ignoring the Ambiguous Case

If

$$\sin B = 0.8,$$

your calculator gives one principal value.
But another angle may satisfy

$$B' = 180^\circ - B.$$

Always check whether the supplementary angle also produces a valid triangle.

Common Error 3: Degrees vs Radians

If

$$\theta = \frac{\pi}{3},$$

then θ is already in radians.
Do not type

$$\sin\left(\frac{\pi}{3}\right)$$

into a calculator set to degree mode.
Similarly, if the problem gives

$$60^\circ,$$

make sure your interpretation is consistent with degrees.

Common Trap

A correct calculation in the wrong angle mode gives a wrong answer.

Common Error 4: Using Degree Formulae With Radian Formulae

The compact formula

$$s = r\theta$$

requires θ in radians.
Likewise,

$$A = \frac{1}{2}r^2\theta$$

requires θ in radians.
If the angle is given in degrees, convert it first.

Common Error 5: Missing Solutions

If

$$\sin x = \frac{1}{2},$$

the answer is not automatically

$$x = 30^\circ.$$

Within $0^\circ < x < 360^\circ$,

$$x = 150^\circ$$

also works.
For wider intervals, periodicity may generate still more solutions.

Common Trap

Always return to the original interval before finalising your answer.

A Better Way to Approach Trigonometry

For a triangle problem:

Label $\rightarrow$ identify known information $\rightarrow$ choose relationship

For a graph problem:

parent graph $\rightarrow$ period $\rightarrow$ symmetry $\rightarrow$ transformation

For an equation:

simplify $\rightarrow$ solve basic trig equation $\rightarrow$ use periodicity $\rightarrow$ check interval

Do Not Overuse the Calculator

A calculator may tell you that

$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ.$$

But it does not automatically explain why

$$150^\circ$$

is also a solution.

Nor does it explain the structure of

$$\sin^2 x + \cos^2 x = 1.$$

TMUA Thinking

Use computation to support mathematical reasoning, not to replace it.

Exact Before Decimal

Whenever possible, retain exact values such as

$$\frac{\sqrt{3}}{2}, \quad \frac{\pi}{3}, \quad 3\sqrt{2}.$$

Avoid replacing them prematurely by decimal approximations.

For example,

$$\sin 60^\circ = \boxed{\frac{\sqrt{3}}{2}}$$

contains exact mathematical information.

A decimal such as

$$0.8660$$

is only an approximation.

Trigonometry Is About Structure

The strongest TMUA solutions recognise relationships such as

$$\sin^2 x + \cos^2 x = 1,$$

$$\tan x = \frac{\sin x}{\cos x},$$

$$\sin x = \sin(\pi - x),$$

and periodic behaviour.

They also recognise when a triangle suggests

$$\frac{1}{2}ab\sin C,$$

the sine rule,
or the cosine rule.

Trigonometry Is About Structure (continued)

Key Idea

Recognition reduces calculation.

Part 5 Formula Map I

Right-triangle ratios:

$$\sin \theta = \frac{O}{H}$$

$$\cos \theta = \frac{A}{H}$$

$$\tan \theta = \frac{O}{A}$$

Triangle area:

$$A = \frac{1}{2}ab \sin C$$

Sine rule:

Part 5 Formula Map I (continued)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Part 5 Formula Map II

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Radians:

$$180^\circ = \pi$$

Arc length:

$$s = r\theta$$

Sector area:

$$A = \frac{1}{2}r^2\theta$$

for θ in radians.

Part 5 Formula Map III

Fundamental identities:

$$\sin^2 \theta + \cos^2 \theta = 1$$

and

$$\tan \theta = \frac{\sin \theta}{\cos \theta}.$$

Periods:

$$\sin x, \cos x : 2\pi$$

$$\tan x : \pi.$$

What You Should Now Be Able to Do

You should now be able to:

- use sine, cosine and tangent;
- derive and recall exact values;
- use $\frac{1}{2}ab \sin C$;
- apply the sine rule;
- recognise the ambiguous case;
- apply the cosine rule;
- convert degrees and radians;
- use arc-length and sector-area formulae;
- interpret sine, cosine and tangent as functions;
- recognise their graphs;
- use symmetry and periodicity;
- understand simple transformations;
- use the two fundamental identities;
- solve trigonometric equations on specified intervals.

Part 5 Self-Check

Can You Explain These Without Looking Back?

- 1 Why does $\tan \theta = \sin \theta / \cos \theta$?
- 2 Where do the exact values for 30° , 45° , 60° come from?
- 3 Why is triangle area $\frac{1}{2}ab \sin C$?
- 4 How can the area formula lead to the sine rule?
- 5 Why can the sine rule give two possible angles?
- 6 Why is the cosine rule a generalisation of Pythagoras?
- 7 What exactly is one radian?
- 8 Why is arc length $r\theta$?
- 9 Why is sector area $\frac{1}{2}r^2\theta$?
- 10 Why does tangent have period π ?
- 11 Why does $\sin^2 x + \cos^2 x = 1$?
- 12 How do you ensure you have found every solution?

TMUA Habits for Trigonometry

- 1 Draw or interpret the geometry first.
- 2 Pair each side with its opposite angle.
- 3 Know exact values.
- 4 Understand radians rather than treating them as a conversion trick.
- 5 Know the three basic trigonometric graphs.
- 6 Use symmetry and periodicity deliberately.
- 7 Convert equations to one trigonometric function when useful.
- 8 Check the ambiguous sine-rule case.
- 9 Keep track of the specified interval.
- 10 Prefer exact answers when possible.

TMUA Thinking

The aim is not formula collection.

It is controlled movement between geometry, functions and algebra.

TMUA Mathematics Masterclass

Part 6: Exponentials and Logarithms

We will move to:

- exponential functions;
- graphs of $y = a^x$;
- logarithms as inverse operations;
- laws of logarithms;
- exponential equations;
- logarithmic equations;
- transformations;
- domains and restrictions;
- TMUA-style exponential and logarithmic reasoning.

Do Not Memorise Trigonometry as a List.

See the geometry, the function and the algebra together.

Geometry → Trigonometric Relationship → Function → Reasoning

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 6: Exponentials and Logarithms

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From Powers to Functions

You already know expressions such as

$$2^3 = 8, \quad 5^{-1} = \frac{1}{5}, \quad 9^{1/2} = 3.$$

In these examples, the exponent is a number.

Now consider

$$y = a^x.$$

The exponent itself becomes the variable.

This produces one of the most important families of functions in mathematics:

exponential functions.

The Central Relationship

Exponentials and logarithms are two ways of expressing the same mathematical relationship.

$$a^b = c$$

means exactly the same thing as

$$\log_a c = b.$$

Thus

$$\text{exponentiation} \longleftrightarrow \text{logarithms}$$

are inverse processes.

Key Idea

A logarithm answers a simple question:

"What exponent produces this number?"

Part 6 Roadmap

We will study:

- 1 Review of index laws
- 2 Exponential functions
- 3 Graphs of $y = a^x$
- 4 The cases $a > 1$, $a = 1$, and $0 < a < 1$
- 5 Exponential growth and decay
- 6 The meaning of a logarithm
- 7 Exponential and logarithmic forms
- 8 Restrictions on logarithms
- 9 Logarithmic graphs
- 10 Exponential and logarithmic functions as inverses
- 11 Product, quotient and power laws
- 12 Special logarithmic values
- 13 Simplifying logarithmic expressions

Part 6 Roadmap (continued)

- 14 Solving exponential equations
- 15 Equations requiring algebraic substitution
- 16 Domain checking
- 17 TMUA-style structural reasoning

Before Logarithms: Know Your Index Laws

For an appropriate non-zero base a ,

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

and

$$a^{-n} = \frac{1}{a^n}.$$

Logarithm laws are closely connected to these index laws.

Fractional Indices

Recall

$$a^{1/n} = \sqrt[n]{a}$$

and therefore

$$a^{m/n} = \sqrt[n]{a^m}.$$

For example,

$$16^{3/4} = \left(16^{1/4}\right)^3 = 2^3 = \boxed{8}.$$

These ideas help explain why exponential functions can be defined beyond integer exponents.

Zero Exponent

For positive a ,

$$a^0 = 1.$$

One algebraic explanation is

$$\frac{a^m}{a^m} = 1.$$

But using index laws,

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

Therefore

$$a^0 = 1.$$

This immediately tells us something important about the graph of $y = a^x$.

The Exponential Function

The general exponential function is

$$y = a^x$$

for a positive base a .

For the main non-constant exponential cases,

$$a > 0, \quad a \neq 1.$$

The value of the base determines the behaviour of the graph.

Every Exponential Graph Passes Through One Point

Since

$$a^0 = 1,$$

putting $x = 0$ into

$$y = a^x$$

gives

$$y = 1.$$

Therefore every positive-base exponential graph passes through

$$(0, 1).$$

This is one of the fastest recognition features of an exponential graph.

Case 1: $a > 1$

Suppose

$$a > 1.$$

Then as x increases,

$$a^x$$

increases.

For example,

$$y = 2^x.$$

Some values are

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

Thus the graph rises from left to right.

Exponential Growth

For

$$a > 1,$$

the function

$$y = a^x$$

shows exponential growth.

It satisfies

$$a^{x+1} = a \cdot a^x.$$

Thus increasing x by 1 multiplies the output by the constant factor a .

Key Idea

Linear growth repeatedly adds.

Exponential growth repeatedly multiplies.

Case 2: $0 < a < 1$

Suppose

$$0 < a < 1.$$

For example,

$$y = \left(\frac{1}{2}\right)^x.$$

Then

x	-2	-1	0	1	2
$(1/2)^x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Thus the graph decreases from left to right.

This is often called

exponential decay.

Growth and Decay Are Closely Connected

Observe that

$$\left(\frac{1}{2}\right)^x = 2^{-x}.$$

Thus

$$y = \left(\frac{1}{2}\right)^x$$

is obtained from

$$y = 2^x$$

by replacing x with $-x$.

Therefore the two graphs are reflections in the y -axis.

More generally,

Growth and Decay Are Closely Connected (continued)

$$\left(\frac{1}{a}\right)^x = a^{-x}.$$

Case 3: $a = 1$

If

$$a = 1,$$

then

$$y = 1^x = 1.$$

Therefore the graph is simply

$$y = 1.$$

This is a constant function rather than a growing or decaying exponential curve.

Why We Restrict the Base

The source considers simple positive values of a .

For negative bases, expressions such as

$$(-2)^x$$

do not produce a real value for every real x .

For example,

$$(-2)^{1/2}$$

is not real.

Thus the familiar continuous real exponential function is naturally considered with

$$a > 0.$$

Core Features of $y = a^x$

For

$$a > 0, \quad a \neq 1,$$

the exponential function has:

$$\text{domain} = \mathbb{R}$$

and

$$\text{range} = (0, \infty).$$

It passes through

$$(0, 1)$$

and never reaches

$$y = 0.$$

Core Features of $y = a^x$ (continued)

Hence

$$y = 0$$

is a horizontal asymptote.

What Happens as the Base Changes?

Compare

$$y = 2^x, \quad y = 3^x, \quad y = 10^x.$$

All pass through

$$(0, 1).$$

For positive x ,

$$10^x$$

grows much faster than

$$3^x,$$

which grows faster than

$$2^x.$$

What Happens as the Base Changes? (continued)

But for negative x , the ordering reverses.

TMUA Thinking

Do not merely memorise the pictures.
Ask how changing a changes the numerical values of a^x .

A Useful Symmetry

Compare

$$y = a^x$$

with

$$y = a^{-x}.$$

Replacing x by $-x$ reflects a graph in the y -axis.
Thus

$$y = a^{-x}$$

is the reflection of

$$y = a^x$$

in the y -axis.
Since

A Useful Symmetry (continued)

$$a^{-x} = \left(\frac{1}{a}\right)^x,$$

growth and decay graphs are naturally paired.

The Meaning of a Logarithm

Suppose

$$2^5 = 32.$$

Then

$$\log_2 32 = 5.$$

Why?

Because the logarithm asks:

“What power of 2 gives 32?”

The answer is

5.

Thus logarithms are fundamentally about finding exponents.

Exponential Form and Logarithmic Form

The relationship

$$a^b = c$$

is equivalent to

$$\log_a c = b.$$

Examples:

$$10^2 = 100 \iff \log_{10} 100 = 2,$$

$$2^5 = 32 \iff \log_2 32 = 5,$$

and

$$3^{-2} = \frac{1}{9} \iff \log_3 \frac{1}{9} = -2.$$

The Base, Argument and Value

In

$$\log_a c = b,$$

we call

a

the **base**,

c

the number whose logarithm is being taken,
and

b

the logarithmic value.

But the most useful interpretation remains

The Base, Argument and Value (continued)

$$a^b = c.$$

A Negative Logarithm Is Perfectly Possible

Consider

$$\log_2 \frac{1}{2}.$$

We ask:

$$2^? = \frac{1}{2}.$$

Since

$$2^{-1} = \frac{1}{2},$$

we obtain

$$\log_2 \frac{1}{2} = -1.$$

Thus logarithmic outputs can be negative.

A Zero Logarithm Is Also Possible

Consider

$$\log_a 1.$$

We ask:

$$a^? = 1.$$

For positive $a \neq 1$,

$$a^0 = 1.$$

Therefore

$$\log_a 1 = 0.$$

This fact will also explain an important point on every logarithmic graph.

What Bases Are Allowed?

For the real logarithm function,

$$a > 0, \quad a \neq 1.$$

Why not $a = 1$?

Because

$$1^x = 1$$

for every x .

Thus 1^x cannot produce arbitrary positive values and cannot have the usual logarithmic inverse.

What Inputs Are Allowed?

For

$$\log_a x,$$

we require

$$x > 0.$$

Therefore expressions such as

$$\log_a 0$$

and

$$\log_a(-5)$$

are not defined in the real logarithmic setting used here.

What Inputs Are Allowed? (continued)

Common Trap

Before manipulating a logarithmic expression, check that every logarithm has a positive argument.

Domain Restrictions Can Control a Problem

Suppose an expression contains

$$\log_2(x - 3).$$

For this logarithm to exist,

$$x - 3 > 0.$$

Hence

$$\boxed{x > 3}.$$

Similarly,

$$\log_5(7 - 2x)$$

requires

$$7 - 2x > 0,$$

Domain Restrictions Can Control a Problem (continued)

so

$$\boxed{x < \frac{7}{2}}.$$

TMUA Thinking

Domain restrictions are part of the mathematics, not an afterthought.

The Logarithm as an Inverse Function

Consider

$$y = 2^x.$$

If

$$x = 3,$$

then

$$y = 8.$$

The exponential function maps

$$3 \mapsto 8.$$

The logarithm reverses this:

$$8 \mapsto 3.$$

The Logarithm as an Inverse Function (continued)

Thus

$$\boxed{\log_2 8 = 3.}$$

Inverse Graphs

The graph of

$$y = \log_2 x$$

is obtained from

$$y = 2^x$$

by interchanging the roles of x and y .

Geometrically, this corresponds to reflection in

$$\boxed{y = x}.$$

Thus

$$\boxed{y = a^x \quad \text{and} \quad y = \log_a x}$$

are inverse-function graphs.

Domain and Range Swap

For

$$y = a^x,$$

the domain is

$$\mathbb{R}$$

and the range is

$$(0, \infty).$$

For the inverse

$$y = \log_a x,$$

these swap:

$$\boxed{\text{domain} = (0, \infty)}$$

Domain and Range Swap (continued)

and

$$\text{range} = \mathbb{R}.$$

This is exactly what we expect from inverse functions.

A Key Point on the Log Graph

Since

$$\log_a 1 = 0,$$

every logarithmic graph passes through

$$(1, 0).$$

Compare this with the exponential graph, which passes through

$$(0, 1).$$

These two points are reflections of each other in

$$y = x.$$

The Logarithmic Asymptote

The exponential graph has horizontal asymptote

$$y = 0.$$

Reflecting this line in

$$y = x$$

produces

$$x = 0.$$

Therefore the logarithmic graph has vertical asymptote

$$x = 0.$$

This agrees with the restriction

$$x > 0.$$

Two Fundamental Inverse Identities

Because exponentials and logarithms undo one another,

$$a^{\log_a x} = x$$

for

$$x > 0.$$

Similarly,

$$\log_a(a^x) = x.$$

These are not mysterious formulae.

They are consequences of the definition of a logarithm.

Example

Simplify

$$2^{\log_2 7}.$$

Since exponentiation base 2 and logarithm base 2 are inverse operations,

$$\boxed{2^{\log_2 7} = 7.}$$

Similarly,

$$\log_5(5^x) = \boxed{x}.$$

The Product Law

The first major law is

$$\boxed{\log_a x + \log_a y = \log_a(xy)}.$$

This corresponds directly to the index law

$$a^p a^q = a^{p+q}.$$

The logarithm law is therefore not an isolated rule.
It reflects the algebra of exponents.

Why Does the Product Law Work?

Let

$$p = \log_a x, \quad q = \log_a y.$$

Then

$$a^p = x, \quad a^q = y.$$

Multiplying,

$$xy = a^p a^q.$$

Using index laws,

$$xy = a^{p+q}.$$

Therefore

$$\log_a(xy) = p + q.$$

Why Does the Product Law Work? (continued)

Hence

$$\log_a(xy) = \log_a x + \log_a y.$$

The Quotient Law

The second major law is

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right).$$

This corresponds to

$$\frac{a^p}{a^q} = a^{p-q}.$$

Again, logarithm laws mirror index laws.

The Power Law

The third major law is

$$k \log_a x = \log_a (x^k).$$

Equivalently,

$$\log_a (x^k) = k \log_a x.$$

This corresponds to the index law

$$(a^p)^k = a^{pk}.$$

The Three Main Laws Together

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^k) = k \log_a x$$

subject, of course, to the logarithms being defined.

Key Idea

Products become sums.

Quotients become differences.

Powers become coefficients.

Special Case: $\log_a a$

Since

$$a^1 = a,$$

we have

$$\log_a a = 1.$$

More generally,

$$\log_a(a^k) = k.$$

For example,

$$\log_3 27 = \log_3(3^3) = \boxed{3}.$$

Special Case: Reciprocal

Using the power law,

$$\frac{1}{x} = x^{-1}.$$

Therefore

$$\log_a \left(\frac{1}{x} \right) = \log_a (x^{-1}).$$

Hence

$$\log_a \left(\frac{1}{x} \right) = -\log_a x.$$

This is another direct consequence of index laws.

Example: Exact Logarithmic Values

Evaluate

$$\log_5 25.$$

Since

$$25 = 5^2,$$

we obtain

$$\log_5 25 = 2.$$

Evaluate

$$\log_7 \sqrt{7}.$$

Since

$$\sqrt{7} = 7^{1/2},$$

Example: Exact Logarithmic Values (continued)

we obtain

$$\log_7 \sqrt{7} = \frac{1}{2}.$$

Combining Logarithms

Simplify

$$\log_a x + \log_a y - \log_a z.$$

First,

$$\log_a x + \log_a y = \log_a(xy).$$

Then

$$\log_a(xy) - \log_a z = \log_a\left(\frac{xy}{z}\right).$$

Therefore

$$\log_a x + \log_a y - \log_a z = \log_a\left(\frac{xy}{z}\right).$$

Expanding a Logarithm

Expand

$$\log_a \left(\frac{x^3 y}{z^2} \right).$$

Using quotient and product laws,

$$= \log_a(x^3) + \log_a y - \log_a(z^2).$$

Then use the power law:

$$3 \log_a x + \log_a y - 2 \log_a z.$$

A Very Common False Rule

It is tempting to write

$$\log_a(x + y) = \log_a x + \log_a y.$$

This is false.

The correct product law is

$$\log_a(xy) = \log_a x + \log_a y.$$

There is no corresponding elementary rule for

$$\log_a(x + y).$$

Common Trap

Logarithms convert multiplication into addition.
They do not convert addition into addition.

Another Common False Rule

Similarly,

$$\log_a(x - y)$$

cannot generally be written as

$$\log_a x - \log_a y.$$

The actual quotient law is

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y.$$

Always distinguish

$$x - y$$

from

Another Common False Rule (continued)

$$\frac{x}{y}.$$

Change of Base — Not Core Specification

Useful, But Not Required by the Specification

The source notes discuss the change-of-base formula,

$$\log_a b = \frac{\log_c b}{\log_c a},$$

but explicitly state that questions requiring knowledge of this formula will not be set. So treat this as mathematical enrichment rather than a core formula to memorise for TMUA.

Why Change of Base Works

Let

$$p = \log_a b.$$

Then

$$a^p = b.$$

Take logarithm base c :

$$\log_c(a^p) = \log_c b.$$

Using the power law,

$$p \log_c a = \log_c b.$$

Therefore

$$p = \frac{\log_c b}{\log_c a}.$$

Why Change of Base Works (continued)

Hence

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

A Reciprocal Identity

Taking $c = b$ gives

$$\log_a b = \frac{\log_b b}{\log_b a}.$$

Since

$$\log_b b = 1,$$

we obtain

$$\log_a b = \frac{1}{\log_b a}.$$

Again, this is useful enrichment rather than essential specification knowledge.

The Core Equation

The source requires equations of the form

$$a^x = b$$

and equations which can be reduced to this form.

If b is an obvious power of a , we can solve immediately.

For example,

$$2^x = 32.$$

Since

$$32 = 2^5,$$

we obtain

$$x = 5.$$

Match the Bases First

Solve

$$9^x = 27.$$

Rewrite both sides using base 3:

$$9 = 3^2, \quad 27 = 3^3.$$

Therefore

$$(3^2)^x = 3^3.$$

So

$$3^{2x} = 3^3.$$

Hence

$$2x = 3$$

Match the Bases First (continued)

and

$$x = \frac{3}{2}.$$

Another Base-Matching Example

Solve

$$4^{x+1} = 8^{2x-1}.$$

Rewrite:

$$4 = 2^2, \quad 8 = 2^3.$$

Thus

$$2^{2(x+1)} = 2^{3(2x-1)}.$$

Equate exponents:

$$2x + 2 = 6x - 3.$$

Hence

$$5 = 4x$$

Another Base-Matching Example (continued)

and

$$x = \frac{5}{4}.$$

What If the Right Side Is Not a Convenient Power?

Consider

$$5^{2x} = 27.$$

We cannot conveniently express 27 as a power of 5.

This is exactly where logarithms become useful.

Take logarithm base 5:

$$\log_5(5^{2x}) = \log_5 27.$$

Therefore

$$2x = \log_5 27.$$

Since

$$27 = 3^3,$$

What If the Right Side Is Not a Convenient Power? (continued)

$$2x = 3 \log_5 3.$$

Hence

$$x = \frac{3}{2} \log_5 3.$$

Exact Answers Matter

For

$$5^{2x} = 27,$$

the expression

$$x = \frac{3}{2} \log_5 3$$

is exact.

A decimal approximation would lose information.

TMUA Thinking

When an exact logarithmic expression is available, do not replace it prematurely with a rounded decimal.

A More Interesting Equation

Consider

$$25^x - 3 \cdot 5^x + 2 = 0.$$

At first this does not look like

$$a^x = b.$$

But observe

$$25^x = (5^2)^x = 5^{2x} = (5^x)^2.$$

Therefore the equation has a hidden quadratic structure.

Substitution Reveals the Quadratic

Let

$$u = 5^x.$$

Then

$$25^x = u^2.$$

So

$$25^x - 3 \cdot 5^x + 2 = 0$$

becomes

$$u^2 - 3u + 2 = 0.$$

Factorise:

$$(u - 1)(u - 2) = 0.$$

Substitution Reveals the Quadratic (continued)

Hence

$$u = 1$$

or

$$u = 2.$$

Return to the Original Variable

Since

$$u = 5^x,$$

the first possibility gives

$$5^x = 1.$$

Therefore

$$x = 0.$$

The second gives

$$5^x = 2.$$

Therefore

$$x = \log_5 2.$$

Return to the Original Variable (continued)

Hence

$$x = 0 \quad \text{or} \quad x = \log_5 2.$$

Recognising Hidden Quadratics

Expressions such as

$$a^{2x} + pa^x + q$$

should make you think

$$u = a^x.$$

Then

$$a^{2x} = (a^x)^2 = u^2.$$

The exponential equation becomes an ordinary quadratic.

Key Idea

Do not attack the exponent first.
Recognise the algebraic structure first.

Example

Solve

$$4^x - 5 \cdot 2^x + 4 = 0.$$

Since

$$4^x = (2^x)^2,$$

let

$$u = 2^x.$$

Then

$$u^2 - 5u + 4 = 0.$$

Factorise:

$$(u - 1)(u - 4) = 0.$$

Example (continued)

Hence

$$u = 1 \quad \text{or} \quad u = 4.$$

Therefore

$$\boxed{x = 0 \text{ or } x = 2}.$$

Remember: a^x Is Positive

For

$$a > 0,$$

we always have

$$a^x > 0.$$

Therefore if a substitution gives

$$u = a^x,$$

then

$$u > 0.$$

This can eliminate algebraic roots.

Example: Rejecting an Impossible Root

Suppose substitution produces

$$u^2 + u - 6 = 0.$$

Then

$$(u + 3)(u - 2) = 0,$$

so algebraically

$$u = -3 \quad \text{or} \quad u = 2.$$

But if

$$u = 5^x,$$

then

$$u > 0.$$

Example: Rejecting an Impossible Root (continued)

Therefore

$$u = -3$$

is impossible.
Only

$$u = 2$$

can correspond to a real x .

Solving a Simple Logarithmic Equation

Consider

$$\log_2 x = 5.$$

Convert to exponential form:

$$x = 2^5.$$

Hence

$$x = 32.$$

This is simply the definition of a logarithm used backwards.

An Equation Using the Product Law

Consider

$$\log_2 x + \log_2(x - 2) = 3.$$

Before doing anything,

$$x > 0$$

and

$$x - 2 > 0.$$

Therefore

$$\boxed{x > 2}.$$

Now combine the logarithms:

$$\log_2[x(x - 2)] = 3.$$

Continue the Equation

From

$$\log_2[x(x - 2)] = 3,$$

convert to exponential form:

$$x(x - 2) = 2^3.$$

Thus

$$x^2 - 2x = 8.$$

So

$$x^2 - 2x - 8 = 0.$$

Factorise:

$$(x - 4)(x + 2) = 0.$$

Continue the Equation (continued)

Hence

$$x = 4 \quad \text{or} \quad x = -2.$$

Domain Check

Our original logarithms required

$$x > 2.$$

Therefore

$$x = -2$$

must be rejected.

The only valid solution is

$$\boxed{x = 4}.$$

Common Trap

Algebra can produce candidates that do not belong to the domain of the original logarithmic expression.

Graphical Interpretation of $a^x = b$

Solving

$$a^x = b$$

can be interpreted as finding the intersection of

$$y = a^x$$

and

$$y = b.$$

For

$$a > 0, \quad a \neq 1, \quad b > 0,$$

the exponential function is one-to-one.

Therefore there is exactly one real solution.

That solution is

Graphical Interpretation of $a^x = b$ (continued)

$$x = \log_a b.$$

Why $a^x = 0$ Has No Real Solution

For every real x ,

$$a^x > 0.$$

The graph may approach the x -axis,
but it never reaches it.

Therefore

$$a^x = 0$$

has no real solution.

This is visible both algebraically and graphically.

Why $a^x = -3$ Has No Real Solution

Again,

$$a^x > 0$$

for positive a .

Therefore

$$a^x = -3$$

is impossible over the reals.

Hence

no real solution.

TMUA Thinking

Sometimes the fastest solution is not calculation.
It is recognising the range of the function.

TMUA Challenge 1

Solve

$$3^{2x} = 27.$$

Since

$$27 = 3^3,$$

we have

$$3^{2x} = 3^3.$$

Therefore

$$2x = 3$$

and

$$x = \frac{3}{2}.$$

TMUA Challenge 1 (continued)

The key recognition was matching the bases.

TMUA Challenge 2

Solve

$$2^{x+1} = 16.$$

Since

$$16 = 2^4,$$

we obtain

$$2^{x+1} = 2^4.$$

Therefore

$$x + 1 = 4.$$

Hence

$$\boxed{x = 3}.$$

TMUA Challenge 3

Evaluate

$$\log_3 \frac{1}{27}.$$

Since

$$27 = 3^3,$$

we have

$$\frac{1}{27} = 3^{-3}.$$

Therefore

$$\boxed{\log_3 \frac{1}{27} = -3}.$$

TMUA Challenge 4

Simplify

$$2 \log_a x + \log_a y.$$

Using the power law,

$$2 \log_a x = \log_a (x^2).$$

Therefore

$$\log_a (x^2) + \log_a y = \log_a (x^2 y).$$

Hence

$$2 \log_a x + \log_a y = \log_a (x^2 y).$$

TMUA Challenge 5

Suppose

$$\log_2 (x - 1) = 3.$$

Convert to exponential form:

$$x - 1 = 2^3.$$

Thus

$$x - 1 = 8$$

and

$$x = 9.$$

Check:

$$x - 1 = 8 > 0,$$

so the logarithm is defined.

TMUA Challenge 6

Solve

$$9^x - 10 \cdot 3^x + 9 = 0.$$

Since

$$9^x = (3^x)^2,$$

let

$$u = 3^x.$$

Then

$$u^2 - 10u + 9 = 0.$$

Hence

$$(u - 1)(u - 9) = 0.$$

TMUA Challenge 6 (continued)

Thus

$$u = 1 \quad \text{or} \quad u = 9.$$

Therefore

$$\boxed{x = 0 \text{ or } x = 2}.$$

TMUA Challenge 7

Without solving numerically, determine whether

$$2^x = -5$$

has a real solution.

Since

$$2^x > 0$$

for every real x ,
it can never equal a negative number.
Therefore

no real solution.

This is a range argument.

TMUA Challenge 8

For what values of x is

$$\log_3(5 - 2x)$$

defined?

We require

$$5 - 2x > 0.$$

Therefore

$$-2x > -5.$$

Dividing by -2 reverses the inequality:

$$x < \frac{5}{2}.$$

Common Error 1: Logarithm of a Sum

Do not write

$$\log_a(x + y) = \log_a x + \log_a y.$$

This is false.

The correct rule is

$$\log_a(xy) = \log_a x + \log_a y.$$

Common Error 2: Ignoring the Domain

An equation may produce

$$x = -2$$

algebraically.

But if the original expression contains

$$\log(x - 1),$$

then

$$x > 1.$$

So $x = -2$ cannot be a valid solution.

Common Trap

Always check candidate solutions against the original logarithmic domain.

Common Error 3: Thinking Logs Must Be Positive

The argument of a real logarithm must be positive.
But the logarithm itself may be

negative,

zero,

or

positive.

For example,

$$\log_2 \frac{1}{2} = -1,$$

$$\log_2 1 = 0,$$

Common Error 3: Thinking Logs Must Be Positive (continued)

$$\log_2 8 = 3.$$

Common Error 4: Confusing Base and Argument

In

$$\log_2 8 = 3,$$

the statement means

$$2^3 = 8.$$

It does not mean

$$8^3 = 2.$$

A reliable way to interpret

$$\log_a c = b$$

is always to rewrite it as

$$a^b = c.$$

Common Error 5: Missing Hidden Quadratics

Consider

$$16^x - 5 \cdot 4^x + 4 = 0.$$

Do not treat 16^x and 4^x as unrelated.

Since

$$16^x = (4^x)^2,$$

the equation is quadratic in

$$4^x.$$

TMUA Thinking

Look for algebraic structure before performing calculations.

A Decision Process for Exponential Equations

When you see an exponential equation, ask:

Can I rewrite both sides with the same base?

If yes,

equate exponents.

If not, ask:

Is there a hidden quadratic in a^x ?

If yes,

$$u = a^x.$$

Otherwise,

use logarithms.

A Decision Process for Logarithms

When you see logarithms:

Step 1: Check the domain

then

Step 2: Look for log laws

then

Step 3: Combine or expand strategically

then

Step 4: Convert between log and exponential form

and finally

Step 5: Check candidate solutions.

Think in Both Directions

You should be able to move instantly between

$$2^5 = 32$$

and

$$\log_2 32 = 5.$$

Likewise,

$$\log_3 81 = 4$$

should immediately tell you

$$3^4 = 81.$$

Key Idea

Fluency means being able to move in both directions without treating them as separate topics.

Use Graphs to Check Reasoning

Suppose

$$2^x = 7.$$

Since

$$2^2 = 4$$

and

$$2^3 = 8,$$

we know immediately that

$$2 < x < 3.$$

Therefore if an algebraic calculation gives something outside this interval, we know something has gone wrong.

Use Graphs to Check Reasoning (continued)

TMUA Thinking

Estimate first when possible.

Exact algebra and graphical intuition should agree.

Part 6 Formula Map I

Index laws:

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^{-n} = \frac{1}{a^n}$$

and

$$a^0 = 1.$$

Part 6 Formula Map II

Exponential-logarithmic equivalence:

$$a^b = c \iff \log_a c = b$$

with

$$a > 0, \quad a \neq 1, \quad c > 0.$$

Special values:

$$\log_a 1 = 0$$

and

$$\log_a a = 1.$$

Part 6 Formula Map III

Logarithm laws:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

$$\log_a(x^k) = k \log_a x$$

and

$$\log_a \left(\frac{1}{x} \right) = -\log_a x.$$

Part 6 Formula Map IV

Inverse relationships:

$$a^{\log_a x} = x$$

and

$$\log_a(a^x) = x.$$

Exponential graph:

$$(0, 1)$$

Logarithmic graph:

$$(1, 0)$$

and the two are reflections in

$$y = x.$$

What You Should Now Be Able to Do

You should now be able to:

- recognise $y = a^x$;
- distinguish exponential growth and decay;
- describe exponential graphs;
- explain why $a^x > 0$;
- interpret logarithms as inverse exponentiation;
- move between exponential and logarithmic forms;
- state the restrictions on real logarithms;
- recognise logarithmic graphs;
- explain the inverse relationship graphically;
- use the three logarithm laws;
- simplify logarithmic expressions;
- solve simple exponential equations;
- recognise hidden quadratics in a^x ;
- use logarithms when bases cannot be matched;
- reject solutions that violate structural constraints.

Part 6 Self-Check

Can You Explain These Without Looking Back?

- ❶ Why does every graph $y = a^x$ pass through $(0, 1)$?
- ❷ Why is a^x always positive for $a > 0$?
- ❸ What changes when $a > 1$ becomes $0 < a < 1$?
- ❹ What exactly does $\log_a b$ mean?
- ❺ Why must the argument of a real logarithm be positive?
- ❻ Why is $\log_a 1 = 0$?
- ❼ Why is $\log_a a = 1$?
- ❽ Why are exponential and logarithmic graphs reflections in $y = x$?
- ❾ Where does the product law for logarithms come from?
- ❿ Why is $\log_a(x + y) \neq \log_a x + \log_a y$?
- ⓫ How do you solve $a^x = b$?
- ⓬ How do you recognise a quadratic hidden inside an exponential equation?

TMUA Habits for Exponentials and Logarithms

- ❶ Know your index laws.
- ❷ Recognise exponential graph structure.
- ❸ Think of logarithms as exponents.
- ❹ Move freely between $a^b = c$ and $\log_a c = b$.
- ❺ Check logarithmic domains immediately.
- ❻ Understand log laws rather than memorising them blindly.
- ❼ Try matching bases before introducing logarithms.
- ❽ Look for substitutions such as $u = a^x$.
- ❾ Remember that $a^x > 0$.
- ⓫ Keep exact logarithmic answers when appropriate.

TMUA Thinking

The important skill is not complicated computation.
It is recognising the structure that makes the computation simple.

Coming in Part 7

TMUA Mathematics Masterclass

Part 7: Differentiation

We will move to:

- gradient as rate of change;
- the derivative;
- tangent gradients;
- derivative notation;
- second derivatives;
- standard differentiation rules;
- product and quotient rules;
- chain rule;

Coming in Part 7 (continued)

- stationary points;
- increasing and decreasing functions;
- optimisation;
- TMUA-style calculus reasoning.

A Logarithm Is an Exponent.

Once that idea is clear, the rules become natural.

$$a^b = c \iff \log_a c = b$$

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 7: Differentiation

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What Is Differentiation Really About?

Differentiation is often introduced as a collection of rules.
But its central idea is much more important:

How rapidly is something changing?

For a graph

$$y = f(x),$$

the derivative tells us how y changes as x changes.

Geometrically, it gives the gradient of the tangent to the curve.

$$f'(x) = \text{instantaneous rate of change} = \text{tangent gradient}$$

Part 7 Roadmap

We will study:

- ① Rate of change
- ② Gradient of a curve
- ③ Tangents and secants
- ④ The derivative
- ⑤ Derivative notation
- ⑥ Second derivatives
- ⑦ Differentiating x^n for rational n
- ⑧ Simplification before differentiation
- ⑨ Tangent equations
- ⑩ Normal equations
- ⑪ Stationary points
- ⑫ Local maxima and minima
- ⑬ Second derivative classification

Part 7 Roadmap (continued)

- 14 Limitations of the second derivative test
- 15 Strictly increasing functions
- 16 Strictly decreasing functions
- 17 Using derivatives to understand graphs
- 18 TMUA-style calculus reasoning

TMUA Scope: Keep the Calculus Focused

TMUA Scope

For this part of TMUA mathematics, the important differentiation technique is

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for rational n , together with sums and differences of powers.
You may first need to simplify an expression into this form.

The focus is therefore not advanced differentiation algorithms.
The focus is:

understanding what derivatives tell us.

Techniques Outside the Required Differentiation Scope

For this TMUA section, you should not build your preparation around techniques such as

product rule,

quotient rule,

chain rule,

or differentiation of trigonometric functions.

Instead, become extremely fluent with simple derivatives and their mathematical interpretation.

TMUA Thinking

Simple calculus can still generate difficult reasoning problems.

What Is a Rate of Change?

A rate of change compares the change in one quantity with the change in another.

For example,

$$\text{speed} = \frac{\text{change in distance}}{\text{change in time}}.$$

If a car travels 60 km in 2 hours, its average speed is

$$\frac{60}{2} = 30 \text{ km/h.}$$

This is an example of an

average rate of change.

Average Rate of Change of a Function

Suppose

$$y = f(x).$$

Between

$$x = a$$

and

$$x = b,$$

the change in y is

$$f(b) - f(a).$$

The change in x is

$$b - a.$$

Average Rate of Change of a Function (continued)

Therefore the average rate of change is

$$\frac{f(b) - f(a)}{b - a}.$$

Geometric Interpretation

The points

$$A = (a, f(a))$$

and

$$B = (b, f(b))$$

lie on the curve.

The gradient of the straight line joining them is

$$\frac{f(b) - f(a)}{b - a}.$$

This line is called a

secant.

Therefore:

Geometric Interpretation (continued)

average rate of change = gradient of a secant.

But What About Change at One Instant?

Average rate of change describes an interval.
But suppose we want the rate of change at exactly

$$x = a.$$

There is only one point.
So we cannot directly calculate

$$\frac{\Delta y}{\Delta x}$$

between two distinct points.
The idea is to move the second point increasingly close to the first.

From Secant to Tangent

Imagine two points on a curve.
As the second point moves closer to the first, the secant line approaches a limiting position.
That limiting line is the

tangent.

Its gradient represents the instantaneous rate of change.
Therefore

derivative at a point = gradient of the tangent.

What Does the Derivative Tell Us?

Suppose

$$y = f(x).$$

Then

$$\frac{dy}{dx}$$

tells us the rate at which y is changing with respect to x .

Equivalently,

$$\frac{dy}{dx} = \text{gradient of the tangent to } y = f(x).$$

The derivative is therefore itself a function of x .

A Function and Its Derivative

Suppose

$$y = x^3 + 7x^2 - 3x + 11.$$

Then

$$\frac{dy}{dx} = 3x^2 + 14x - 3.$$

The original function tells us the height of the curve.

Given x ,

$$f(x)$$

tells us the corresponding y -coordinate.

The derivative instead tells us

$$\text{the gradient of the curve at that } x.$$

Two Functions, Two Different Jobs

If

$$f(x) = x^3 + 7x^2 - 3x + 11,$$

then

$$f(2)$$

tells us the height of the curve when $x = 2$.

But

$$f'(2)$$

tells us the gradient of the tangent when $x = 2$.

Thus

$$f(x) \rightarrow \text{position on the graph}$$

while

Two Functions, Two Different Jobs (continued)

$$f'(x) \rightarrow \text{local behaviour of the graph.}$$

Notation You Must Recognise

If

$$y = f(x),$$

the first derivative may be written as

$$\frac{dy}{dx}$$

or

$$f'(x).$$

The second derivative may be written as

$$\frac{d^2y}{dx^2}$$

or

Notation You Must Recognise (continued)

$$f''(x).$$

You should move comfortably between these notations.

What Is the Second Derivative?

The first derivative tells us how y changes.

The second derivative tells us how the first derivative changes.

Thus

$$f'(x)$$

describes the gradient,
while

$$f''(x)$$

describes how that gradient is changing.

Key Idea

The second derivative contains information about the changing shape of the curve.

A Simple Example

Let

$$y = x^3.$$

Then

$$\frac{dy}{dx} = 3x^2.$$

Differentiate again:

$$\frac{d^2y}{dx^2} = 6x.$$

Thus

$$f(x) = x^3, \quad f'(x) = 3x^2, \quad f''(x) = 6x.$$

Each describes a different level of change.

The Power Rule

The central differentiation rule required here is

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for rational n .

In words:

multiply by the old power, then reduce the power by one.

Example: Positive Integer Power

Differentiate

$$y = x^5.$$

Using

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

we obtain

$$\frac{dy}{dx} = 5x^4.$$

Similarly,

$$\frac{d}{dx}(x^8) = 8x^7.$$

Constant Multiples

Differentiate

$$y = 7x^4.$$

The constant 7 remains as a multiplier:

$$\frac{dy}{dx} = 7(4x^3).$$

Therefore

$$\boxed{\frac{dy}{dx} = 28x^3.}$$

Sums and Differences

Differentiate

$$y = 3x^4 - 5x^3 + 7x^2 - 9x + 4.$$

Differentiate each term:

$$\frac{dy}{dx} = 12x^3 - 15x^2 + 14x - 9.$$

The derivative of the constant 4 is zero.
Thus

$$\boxed{\frac{dy}{dx} = 12x^3 - 15x^2 + 14x - 9.}$$

Why Does a Constant Differentiate to Zero?

Consider

$$y = 7.$$

Its graph is a horizontal line.
A horizontal line has gradient

$$0.$$

Therefore

$$\frac{d}{dx}(7) = 0.$$

This is not merely a rule to memorise.
It follows directly from the geometric meaning of a derivative.

Negative Powers

The power rule also applies to negative rational powers.
For example,

$$y = x^{-2}.$$

Then

$$\frac{dy}{dx} = -2x^{-3}.$$

Therefore

$$\frac{dy}{dx} = -\frac{2}{x^3}.$$

Fractional Powers

Consider

$$y = x^{1/2}.$$

Then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}.$$

Hence

$$\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{x}}}$$

where the expression is defined.

Another Rational-Power Example

Differentiate

$$y = x^{3/2}.$$

Then

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}.$$

Therefore

$$\boxed{\frac{dy}{dx} = \frac{3}{2}\sqrt{x}.$$

The same power rule works.

A Very Important TMUA Habit

Some expressions may initially look as though they require advanced differentiation rules. But often they can first be simplified.

TMUA Thinking

Before differentiating, ask:

Can I simplify this into powers of x ?

This is particularly important for TMUA.

Example: Simplify First

Suppose

$$y = \frac{(3x + 2)^2}{x^{1/2}}.$$

Do not immediately search for a quotient rule.

First expand:

$$(3x + 2)^2 = 9x^2 + 12x + 4.$$

Therefore

$$y = \frac{9x^2 + 12x + 4}{x^{1/2}}.$$

Rewrite as Powers

Now divide each term by $x^{1/2}$:

$$y = 9x^{3/2} + 12x^{1/2} + 4x^{-1/2}.$$

Now every term is simply a power of x .
Therefore

$$\frac{dy}{dx} = \frac{27}{2}x^{1/2} + 6x^{-1/2} - 2x^{-3/2}.$$

The Structural Lesson

The expression

$$\frac{(3x + 2)^2}{x^{1/2}}$$

looks more complicated than

$$9x^{3/2} + 12x^{1/2} + 4x^{-1/2}.$$

But they are algebraically identical.
The second form makes the differentiation immediate.

Key Idea

Good algebra often removes the need for more complicated calculus.

Finding the Gradient at a Point

Suppose

$$y = x^3 - 2x^2 + 4x + 1.$$

Then

$$\frac{dy}{dx} = 3x^2 - 4x + 4.$$

At

$$x = 2,$$

the gradient is

$$3(2)^2 - 4(2) + 4.$$

Thus

$$12 - 8 + 4 = 8.$$

Finding the Gradient at a Point (continued)

Therefore the tangent gradient is

$$\boxed{8}.$$

Derivative Value vs Coordinate

Be careful:

$$f'(2) = 8$$

does not mean that the point is

$$(2, 8).$$

The number 8 is the gradient.

To find the point on the curve, calculate

$$f(2).$$

Thus:

$$f(a) = \text{height}$$

and

Derivative Value vs Coordinate (continued)

$$f'(a) = \text{gradient.}$$

Equation of a Tangent

Suppose we want the tangent to

$$y = f(x)$$

at

$$x = a.$$

We need two things:

point

and

gradient.

The point is

$$(a, f(a)).$$

Equation of a Tangent (continued)

The gradient is

$$f'(a).$$

Then use the straight-line equation

$$y - y_1 = m(x - x_1).$$

Example: Tangent to a Curve

Find the tangent to

$$y = x^2 + 1$$

at

$$x = 2.$$

First find the point:

$$y = 2^2 + 1 = 5.$$

So the point is

$$(2, 5).$$

Differentiate:

$$\frac{dy}{dx} = 2x.$$

Example: Tangent to a Curve (continued)

At $x = 2$,

$$m = 4.$$

Complete the Tangent Equation

Using

$$y - y_1 = m(x - x_1),$$

we obtain

$$y - 5 = 4(x - 2).$$

Therefore

$$y - 5 = 4x - 8$$

and hence

$$y = 4x - 3.$$

What Is a Normal?

The normal to a curve at a point is the straight line perpendicular to the tangent at that point.
If the tangent gradient is

$$m,$$

then, when $m \neq 0$, the normal gradient is

$$-\frac{1}{m}.$$

This follows from the perpendicular-gradient relationship

$$m_1 m_2 = -1.$$

Example: Normal to $y = x^2 + 1$

At $x = 2$, we found that the tangent gradient is

$$4.$$

Therefore the normal gradient is

$$-\frac{1}{4}.$$

The point is

$$(2, 5).$$

Hence

$$y - 5 = -\frac{1}{4}(x - 2).$$

This is the equation of the normal.

What Is a Stationary Point?

At a stationary point, the tangent is horizontal.

A horizontal tangent has gradient

$$0.$$

Therefore stationary points occur where

$$\frac{dy}{dx} = 0.$$

Equivalently,

$$f'(x) = 0.$$

Stationary Does Not Automatically Mean Maximum

A stationary point may be a

local maximum

or a

local minimum.

More generally, a horizontal point of inflexion can also be stationary.

However, the examined stationary-point classifications in this TMUA section are maxima and minima.

Example: Finding Stationary Points

Consider

$$y = 2x^3 - 3x^2 - 12x + 6.$$

Differentiate:

$$\frac{dy}{dx} = 6x^2 - 6x - 12.$$

Factorise:

$$\frac{dy}{dx} = 6(x^2 - x - 2)$$

so

$$\frac{dy}{dx} = 6(x - 2)(x + 1).$$

Stationary x-Coordinates

Set

$$\frac{dy}{dx} = 0.$$

Then

$$6(x - 2)(x + 1) = 0.$$

Therefore

$$x = 2$$

or

$$x = -1.$$

These are the x-coordinates of the stationary points.

To obtain the complete coordinates, substitute them into the original function.

Coordinates of the Stationary Points

For

$$y = 2x^3 - 3x^2 - 12x + 6,$$

at $x = 2$,

$$y = 2(8) - 3(4) - 24 + 6 = -14.$$

So one stationary point is

$$(2, -14).$$

At $x = -1$,

$$y = -2 - 3 + 12 + 6 = 13.$$

Thus the other is

$$(-1, 13).$$

Method 1: Use the Shape of the Curve

For

$$y = 2x^3 - 3x^2 - 12x + 6,$$

the leading term is

$$2x^3.$$

It is a positive cubic.

Therefore its broad end behaviour is

$$y \rightarrow -\infty \quad \text{as} \quad x \rightarrow -\infty,$$

and

$$y \rightarrow +\infty \quad \text{as} \quad x \rightarrow +\infty.$$

This helps us interpret the two stationary points.

Using Global Shape Intelligently

For a positive cubic with two turning points,
the left stationary point is the local maximum,
and the right stationary point is the local minimum.
Since

$$-1 < 2,$$

we therefore expect

$(-1, 13)$ to be a local maximum

and

$(2, -14)$ to be a local minimum.

This can be confirmed using derivatives.

Method 2: Look at the Sign of $f'(x)$

At a local minimum, the graph typically changes from

decreasing

to

increasing.

Thus $f'(x)$ changes

$$- \rightarrow +.$$

At a local maximum, it changes from increasing to decreasing:

$$+ \rightarrow -.$$

This is often called a first-derivative sign test.

Sign Analysis for the Cubic

We have

$$f'(x) = 6(x - 2)(x + 1).$$

Critical values:

$$-1, \quad 2.$$

Check the intervals:

$$x < -1, \quad -1 < x < 2, \quad x > 2.$$

The derivative signs are

$$+ \quad | \quad - \quad | \quad +.$$

Therefore:

$$x = -1 : \quad + \rightarrow -$$

Sign Analysis for the Cubic (continued)

so it is a local maximum,
while

$$x = 2: \quad - \rightarrow +$$

so it is a local minimum.

The Second Derivative at a Stationary Point

Suppose

$$f'(a) = 0.$$

If

$$f''(a) > 0,$$

then the stationary point is a local minimum.

If

$$f''(a) < 0,$$

then the stationary point is a local maximum.

These are extremely useful sufficient conditions.

Why Does $f''(a) > 0$ Suggest a Minimum?

Near a typical minimum,
the first derivative changes from

negative

to

positive.

Thus $f'(x)$ is increasing through zero.
The second derivative measures how $f'(x)$ changes.
Therefore

$$f''(a) > 0$$

is consistent with a minimum.

Why Does $f''(a) < 0$ Suggest a Maximum?

Near a typical maximum,
the first derivative changes from

positive

to

negative.

Thus $f'(x)$ is decreasing through zero.
Therefore

$$f''(a) < 0$$

indicates a local maximum.

Apply the Second Derivative Test

For

$$f(x) = 2x^3 - 3x^2 - 12x + 6,$$

we found

$$f'(x) = 6x^2 - 6x - 12.$$

Differentiate again:

$$f''(x) = 12x - 6.$$

At

$$x = -1,$$

$$f''(-1) = -18 < 0.$$

Therefore

Apply the Second Derivative Test (continued)

$x = -1$ gives a local maximum.

Classify the Other Point

At

$$x = 2,$$

$$f''(2) = 24 - 6 = 18 > 0.$$

Therefore

$x = 2$ gives a local minimum.

This agrees with our previous reasoning.

Are These Conditions Necessary?

We know that

$$f'(a) = 0, \quad f''(a) > 0$$

is sufficient for a local minimum.

And

$$f'(a) = 0, \quad f''(a) < 0$$

is sufficient for a local maximum.

But these conditions are

not necessary.

This distinction is important.

Example: $y = x^4$

Consider

$$y = x^4.$$

Then

$$y' = 4x^3$$

and

$$y'' = 12x^2.$$

At

$$x = 0,$$

we have

$$y'(0) = 0$$

Example: $y = x^4$ (continued)

and

$$y''(0) = 0.$$

Yet $x = 0$ is clearly a minimum.

What Does This Show?

The condition

$$f''(a) > 0$$

is sufficient for identifying a minimum at a stationary point.
But a minimum can exist even when

$$f''(a) = 0.$$

Thus

$$f'(a) = 0, \quad f''(a) > 0$$

is not a necessary condition for a minimum.

Example: $y = -x^4$

Consider

$$y = -x^4.$$

Then

$$y' = -4x^3$$

and

$$y'' = -12x^2.$$

At

$$x = 0,$$

$$y'(0) = 0, \quad y''(0) = 0.$$

Nevertheless,

Example: $y = -x^4$ (continued)

$x = 0$ is a local maximum.

The Logical Lesson

Therefore:

$$f'(a) = 0, \quad f''(a) > 0$$

guarantees a local minimum.
But

local minimum $\nRightarrow f''(a) > 0$.

Similarly,

$$f'(a) = 0, \quad f''(a) < 0$$

guarantees a local maximum,
but

local maximum $\nRightarrow f''(a) < 0$.

What If $f''(a) = 0$?

If

$$f'(a) = 0$$

and

$$f''(a) = 0,$$

the second derivative test does not settle the classification.
You need another argument.

For example:

- inspect the sign of $f'(x)$ around a ;
- use the known shape of the curve;
- compare nearby function values when appropriate.

Common Trap

$f''(a) = 0$ does not automatically mean a point of inflexion.

What Does Increasing Mean Visually?

Imagine moving from left to right along a graph.

If the graph continually rises, we think of the function as increasing.

If it continually falls, we think of it as decreasing.

Differentiation provides a way to connect this visual behaviour with algebra.

Strictly Increasing: TMUA Condition

The specification uses the condition

$$f'(x) > 0$$

to conclude that the function is strictly increasing.

Why?

Because a positive derivative means that the tangent gradient is positive.

Thus the curve is locally sloping upwards.

Strictly Decreasing: TMUA Condition

Similarly,

$$f'(x) < 0$$

implies that the function is strictly decreasing.

A negative derivative means that the tangent gradient is negative.

Thus the graph is locally sloping downwards.

A Logical Warning

The source is careful about the direction of implication.

We may use

$$f'(x) > 0 \implies f \text{ is strictly increasing}$$

and

$$f'(x) < 0 \implies f \text{ is strictly decreasing.}$$

These are sufficient conditions in the framework used here.

Do not automatically reverse these implications.

Why Can the Reverse Direction Be Dangerous?

Consider

$$f(x) = x^3.$$

This function is strictly increasing on the real numbers.

But

$$f'(x) = 3x^2.$$

At

$$x = 0,$$

$$f'(0) = 0.$$

Therefore a strictly increasing function need not satisfy

$$f'(x) > 0$$

at every single point.

Another Subtlety: Corners

A graph may also have a corner.
At a corner, there is no unique tangent gradient.
Therefore

$$f'(x)$$

may not exist there.
Yet the function may still continue increasing through the corner.

TMUA Thinking

Do not confuse a convenient derivative condition with the full mathematical definition of increasing behaviour.

Example

Consider

$$f(x) = x^3 - 3x.$$

Then

$$f'(x) = 3x^2 - 3.$$

Factorise:

$$f'(x) = 3(x^2 - 1)$$

so

$$f'(x) = 3(x - 1)(x + 1).$$

The derivative is zero at

$$x = -1, \quad x = 1.$$

Sign of the Derivative

Consider the intervals

$$x < -1, \quad -1 < x < 1, \quad x > 1.$$

For

$$x < -1,$$

$$f'(x) > 0.$$

For

$$-1 < x < 1,$$

$$f'(x) < 0.$$

For

Sign of the Derivative (continued)

$$x > 1,$$

$$f'(x) > 0.$$

Interpret the Signs

Therefore the curve is strictly increasing on

$$(-\infty, -1)$$

and

$$(1, \infty)$$

using the derivative condition.

It is strictly decreasing on

$$(-1, 1).$$

The derivative has converted a graph-shape problem into a sign problem.

A Powerful Representation

Suppose

$$f'(x) = (x + 2)(x - 3).$$

Then

$$f'(x) = 0$$

at

$$x = -2, \quad x = 3.$$

A sign diagram might be written as

x		-2		3	
$f'(x)$	$+$	0	$-$	0	$+$

This tells us much of the graph's behaviour immediately.

Reading the Sign Diagram

From

$$+ \quad 0 \quad - \quad 0 \quad +,$$

we see:

at $x = -2$,

$$+ \rightarrow -$$

so there is a local maximum.

At $x = 3$,

$$- \rightarrow +$$

so there is a local minimum.

This is often faster than calculating many function values.

TMUA-Style Tangent Question

Suppose

$$f(x) = x^3 - 4x + 2.$$

Find where the tangent has gradient 8.

Differentiate:

$$f'(x) = 3x^2 - 4.$$

We require

$$3x^2 - 4 = 8.$$

Therefore

$$3x^2 = 12.$$

Hence

TMUA-Style Tangent Question (continued)

$$x^2 = 4.$$

So

$$x = \pm 2.$$

A Useful Interpretation

The question

“Where does the tangent have gradient m ?”

translates directly into

$$f'(x) = m.$$

This is a common conceptual bridge:

$$\text{geometry} \longleftrightarrow \text{derivative equation.}$$

Parallel Tangents

Suppose a line has gradient

$$m.$$

A tangent to

$$y = f(x)$$

is parallel to that line when

$$f'(x) = m.$$

Thus a geometric parallelism condition becomes an algebraic equation involving the derivative.

Perpendicular Tangents

If a line has non-zero gradient m , a perpendicular line has gradient

$$-\frac{1}{m}.$$

Therefore a tangent to $y = f(x)$ perpendicular to a given line can be found by solving

$$f'(x) = -\frac{1}{m}.$$

Again, differentiation connects curve geometry with straight-line geometry.

Why Stationary Points Matter

Many problems ask us to maximise or minimise a quantity.

Examples include:

- maximum area;
- minimum distance;
- maximum volume;
- minimum cost;
- greatest or least value of an expression.

If the quantity is represented by a differentiable function, stationary points often provide the relevant candidates.

General Optimisation Strategy

Suppose a quantity Q depends on x .

A typical strategy is:

1. Construct $Q(x)$

2. Differentiate

3. Solve $Q'(x) = 0$

4. Classify/check the candidate

5. Interpret in the original context.

Optimisation Example

Suppose

$$A(x) = 12x - x^2.$$

Find its maximum value.
Differentiate:

$$A'(x) = 12 - 2x.$$

At a stationary point,

$$12 - 2x = 0.$$

Therefore

$$x = 6.$$

Classify the Stationary Point

Differentiate again:

$$A''(x) = -2.$$

Since

$$A''(6) < 0,$$

the stationary point is a maximum.
Now

$$A(6) = 72 - 36 = 36.$$

Therefore

$$\text{maximum value} = 36.$$

Differentiation as a Graph-Sketching Tool

Differentiation helps us answer:

- Where is the graph rising?
- Where is it falling?
- Where are the horizontal tangents?
- Which stationary points are maxima?
- Which are minima?

Thus differentiation is not just an algebraic procedure.
It is a tool for understanding

the geometry of functions.

The First Derivative as a Behaviour Map

If

$$f'(x) > 0,$$

the curve is locally rising.

If

$$f'(x) < 0,$$

the curve is locally falling.

If

$$f'(x) = 0,$$

the tangent is horizontal.

Thus the sign of $f'(x)$ provides a compact description of the graph's behaviour.

The Second Derivative Adds More Information

The second derivative describes how the gradient changes.

At a stationary point:

$$f''(x) > 0$$

is sufficient for a local minimum,
while

$$f''(x) < 0$$

is sufficient for a local maximum.
Thus

$$f'(x)$$

tells us about slope,
and

$$f''(x)$$

The Second Derivative Adds More Information (continued)

helps tell us how that slope is changing.

What About Points of Inflexion?

The supplied specification states that points of inflexion are not examined as part of the stationary-point applications here.

However, a qualitative understanding of their appearance for simple polynomial curves is expected.

So you should recognise that a curve can change its bending behaviour without treating inflexion-point calculation as the main focus of this section.

A Useful Warning

A stationary point does not have to be a maximum or minimum in general mathematics.

For example,

$$y = x^3$$

has

$$y' = 3x^2.$$

Thus

$$y'(0) = 0.$$

But the graph continues increasing through the origin.

This illustrates why

$$f'(a) = 0$$

alone does not classify a stationary point.

TMUA Challenge 1

Differentiate

$$f(x) = 4x^5 - 3x^3 + 7x - 2.$$

We obtain

$$f'(x) = 20x^4 - 9x^2 + 7.$$

Notice that the constant disappears.

TMUA Challenge 2

Differentiate

$$f(x) = 3x^{1/2} + 2x^{-2}.$$

Using the power rule,

$$f'(x) = \frac{3}{2}x^{-1/2} - 4x^{-3}.$$

Therefore

$$f'(x) = \frac{3}{2\sqrt{x}} - \frac{4}{x^3}.$$

TMUA Challenge 3

For

$$f(x) = x^3 - 6x^2 + 9x,$$

find the stationary x -coordinates.

Differentiate:

$$f'(x) = 3x^2 - 12x + 9.$$

Factorise:

$$f'(x) = 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3).$$

Therefore

$$x = 1, 3.$$

TMUA Challenge 4

Classify the stationary points from the previous example.

We have

$$f''(x) = 6x - 12.$$

At

$$x = 1,$$

$$f''(1) = -6 < 0.$$

Therefore $x = 1$ gives a local maximum.

At

$$x = 3,$$

$$f''(3) = 6 > 0.$$

Therefore $x = 3$ gives a local minimum.

TMUA Challenge 5

Suppose

$$f'(x) = (x - 2)(x + 5).$$

Without knowing $f(x)$, determine where f is increasing.

The roots are

$$x = -5, \quad x = 2.$$

The product is positive when

$$x < -5$$

or

$$x > 2.$$

Therefore, using the derivative condition,

TMUA Challenge 5 (continued)

f is strictly increasing for $x < -5$ and $x > 2$.

TMUA Challenge 6

Suppose

$$f'(x) = -(x - 1)(x - 4).$$

Determine the nature of the stationary points.

For

$$x < 1,$$

$$f'(x) < 0.$$

For

$$1 < x < 4,$$

$$f'(x) > 0.$$

For

$$x > 4,$$

TMUA Challenge 6 (continued)

$$f'(x) < 0.$$

Thus

$$x = 1 : - \rightarrow +$$

is a minimum,
and

$$x = 4 : + \rightarrow -$$

is a maximum.

TMUA Challenge 7

A curve has

$$f'(3) = 0$$

and

$$f''(3) = 5.$$

What can we conclude?

Since

$$f'(3) = 0,$$

$x = 3$ is stationary.

Since

$$f''(3) > 0,$$

we can conclude

TMUA Challenge 7 (continued)

$x = 3$ is a local minimum.

TMUA Challenge 8

A curve has

$$f'(2) = 0$$

and

$$f''(2) = 0.$$

Can we conclude that $x = 2$ is not a maximum or minimum?

No.

For example,

$$y = x^4$$

has a minimum where both the first and second derivatives are zero.
More information is required.

Common Error 1: Confusing $f(x)$ and $f'(x)$

Remember:

$$f(a)$$

gives the y -coordinate.
But

$$f'(a)$$

gives the gradient.
These numbers answer completely different questions.

Common Error 2: Solving $f(x) = 0$

To find stationary points, solve

$$f'(x) = 0,$$

not

$$f(x) = 0.$$

The equation

$$f(x) = 0$$

finds x -intercepts.

The equation

$$f'(x) = 0$$

finds horizontal tangents.

Common Error 3: Forgetting the Coordinates

Solving

$$f'(x) = 0$$

usually gives the x -coordinate of a stationary point.

If the question asks for the point itself, you must also find

$$y = f(x).$$

So always distinguish

stationary value of x

from

stationary point (x, y) .

Common Error 4: Overusing the Second Derivative Test

If

$$f''(a) > 0,$$

you can classify the stationary point as a minimum.

If

$$f''(a) < 0,$$

you can classify it as a maximum.

But if

$$f''(a) = 0,$$

do not force a conclusion.

Use another method.

Common Error 5: Reversing an Implication

The TMUA condition

$$f'(x) > 0 \implies f \text{ strictly increasing}$$

does not justify blindly writing

$$f \text{ strictly increasing} \implies f'(x) > 0$$

at every point.

For example,

$$f(x) = x^3$$

is strictly increasing but

$$f'(0) = 0.$$

Common Error 6: Using Unnecessary Calculus Rules

If an expression can be rewritten as

$$3x^{5/2} + 7x^{-1/2} - 4x,$$

differentiate that form directly.

Do not introduce advanced rules unnecessarily.

TMUA Thinking

Simplification is often part of the problem.

The Differentiation Decision Process

When you see a differentiation problem:

1. Simplify first

2. Rewrite roots/fractions as powers if useful

3. Differentiate

4. Ask what the derivative means

5. Use signs, zeros and values intelligently.

The final interpretation is often more important than the differentiation itself.

Translate Words into Derivative Conditions

Learn these translations:

$$\text{horizontal tangent} \implies f'(x) = 0$$

$$\text{tangent gradient } m \implies f'(x) = m$$

$$f'(x) > 0 \implies \text{strictly increasing}$$

$$f'(x) < 0 \implies \text{strictly decreasing.}$$

Stationary-Point Decision Process

Once you have found

$$f'(a) = 0,$$

ask:

What happens to the sign of $f'(x)$?

or calculate

$$f''(a).$$

If

$$f''(a) > 0,$$

minimum.

If

Stationary-Point Decision Process (continued)

$$f''(a) < 0,$$

maximum.

If

$$f''(a) = 0,$$

investigate further.

Think Graphically

When differentiating, continually imagine the graph.

If

$$f'(x) > 0,$$

visualise an upward-sloping tangent.

If

$$f'(x) < 0,$$

visualise a downward-sloping tangent.

If

$$f'(x) = 0,$$

visualise a horizontal tangent.

This makes differentiation a geometric language rather than only an algebraic algorithm.

Part 7 Formula Map I

Derivative:

$$f'(x) = \text{gradient of tangent} = \text{instantaneous rate of change.}$$

Power rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for rational n .

Second derivative:

$$f''(x) = \frac{d}{dx}(f'(x)).$$

Part 7 Formula Map II

Stationary points:

$$f'(x) = 0.$$

At a stationary point $x = a$,

$$f''(a) > 0 \implies \text{local minimum}$$

and

$$f''(a) < 0 \implies \text{local maximum.}$$

Part 7 Formula Map III

Increasing:

$$f'(x) > 0 \implies f \text{ strictly increasing.}$$

Decreasing:

$$f'(x) < 0 \implies f \text{ strictly decreasing.}$$

First derivative sign changes:

$$+ \rightarrow - \implies \text{local maximum}$$

$$- \rightarrow + \implies \text{local minimum.}$$

Part 7 Formula Map IV

Tangent at $x = a$:

$$m_{\text{tangent}} = f'(a).$$

Point of contact:

$$(a, f(a)).$$

Tangent equation:

$$y - f(a) = f'(a)(x - a).$$

For non-zero tangent gradient,

$$m_{\text{normal}} = -\frac{1}{f'(a)}.$$

What You Should Now Be Able to Do

You should now be able to:

- interpret a derivative as a rate of change;
- interpret it as a tangent gradient;
- distinguish $f(x)$ from $f'(x)$;
- recognise first- and second-derivative notation;
- differentiate rational powers of x ;
- simplify expressions before differentiating;
- find tangent gradients;
- find equations of tangents;
- find equations of normals;
- locate stationary points;
- classify maxima and minima;
- use derivative sign changes;
- use the second derivative test correctly;
- understand its logical limitation;
- identify intervals of increasing/decreasing behaviour;
- use differentiation to reason about graph shape.

Part 7 Self-Check

Can You Explain These Without Looking Back?

- 1 What does $f'(a)$ mean geometrically?
- 2 What does it mean as a rate of change?
- 3 What is the difference between $f(a)$ and $f'(a)$?
- 4 Why does a stationary point satisfy $f'(a) = 0$?
- 5 Why does $f''(a) > 0$ identify a minimum when $f'(a) = 0$?
- 6 Why is $f''(a) > 0$ not necessary for a minimum?
- 7 What should you do if $f''(a) = 0$?
- 8 Why does $f'(x) > 0$ imply increasing behaviour?
- 9 Why should that implication not simply be reversed?
- 10 How do you find a tangent equation?
- 11 How do you find a normal gradient?
- 12 Why should you simplify before differentiating?

TMUA Habits for Differentiation

- 1 Understand derivatives geometrically.
- 2 Connect gradient with rate of change.
- 3 Simplify before differentiating.
- 4 Be fluent with rational powers.
- 5 Translate geometric statements into derivative equations.
- 6 Use $f'(x) = 0$ to locate stationary points.
- 7 Use signs of $f'(x)$ to understand graph behaviour.
- 8 Use $f''(x)$ carefully.
- 9 Distinguish sufficient from necessary conditions.
- 10 Keep connecting calculus back to the shape of the graph.

TMUA Thinking

The derivative calculation is often the easy part.
The real question is what the derivative tells you.

Coming in Part 8

TMUA Mathematics Masterclass

Part 8: Integration

We will move to:

- integration as reverse differentiation;
- indefinite integrals;
- the constant of integration;
- integration of powers;
- definite integrals;
- signed area;
- area between curves and axes;
- the Fundamental Theorem of Calculus;

- combining integrals;
- the trapezium rule;
- simple differential equations;
- TMUA-style integration reasoning.

Differentiation Describes Change.

A derivative is not merely an expression to calculate.

It tells us how a function is behaving.

$$f'(x) = \text{gradient} = \text{rate of change}$$

Mathematics Elevate Academy

TMUA Mathematics Masterclass

Part 8: Integration

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From Differentiation to Integration

In Part 7, we studied differentiation.
For example,

$$\frac{d}{dx} \left(\frac{x^3}{3} \right) = x^2.$$

Integration asks us to travel in the opposite direction:

$$x^2 \longrightarrow \frac{x^3}{3} + C.$$

So one important interpretation is

integration is reverse differentiation.

But integration has another important interpretation.
It is connected with the region between a graph and an axis.

Two Ways to Think About Integration

There are two central viewpoints.

Viewpoint 1

Reverse Differentiation

Given $f(x)$, find a function $F(x)$ such that

$$F'(x) = f(x).$$

Viewpoint 2

Signed Area

A definite integral measures accumulated contribution between a curve and an axis.

These ideas are connected through the Fundamental Theorem of Calculus.

Part 8 Roadmap

We will study:

- 1 Integration as reverse differentiation
- 2 Indefinite integrals
- 3 Why we need the constant C
- 4 Integrating powers
- 5 Simplification before integration
- 6 Definite integrals
- 7 Signed area versus geometric area
- 8 Symmetry and cancellation
- 9 The Fundamental Theorem of Calculus
- 10 Reversing limits
- 11 Splitting intervals
- 12 Combining integrals
- 13 The trapezium rule

Part 8 Roadmap (continued)

- 14 Overestimates and underestimates
- 15 Simple differential equations
- 16 Conditions that determine the constant
- 17 TMUA-style integration reasoning

What Does an Indefinite Integral Ask?

Consider

$$\int x^2 dx.$$

A useful way to read this is:

What function differentiates to x^2 ?

Since

$$\frac{d}{dx} \left(\frac{x^3}{3} \right) = x^2,$$

we have

$$\int x^2 dx = \frac{x^3}{3} + C.$$

Why Do We Add C ?

Suppose

$$F'(x) = f(x).$$

Then

$$(F(x) + 7)' = f(x),$$

because

$$\frac{d}{dx}(7) = 0.$$

Likewise,

$$(F(x) - 100)' = f(x).$$

In fact,

$$(F(x) + C)' = f(x)$$

Why Do We Add C ? (continued)

for every constant C .

Therefore

$$\int f(x) dx = F(x) + C.$$

The Family of Antiderivatives

For example,

$$\int 2x \, dx = x^2 + C.$$

This represents infinitely many functions:

$$x^2,$$

$$x^2 + 1,$$

$$x^2 - 5,$$

$$x^2 + 100,$$

and so on.

They all have the same derivative:

The Family of Antiderivatives (continued)

$$2x.$$

Geometrically, they are vertical translations of one another.

Differentiation Loses Constants

Suppose

$$y = x^3 + 17.$$

Then

$$\frac{dy}{dx} = 3x^2.$$

But

$$y = x^3 - 50$$

also gives

$$\frac{dy}{dx} = 3x^2.$$

Differentiation destroys information about additive constants.
Integration therefore restores that uncertainty through

Differentiation Loses Constants (continued)

$$\boxed{+C}.$$

The Power Rule for Integration

We know

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n.$$

Therefore, reversing differentiation,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

provided

$$n \neq -1.$$

The exceptional $n = -1$ case is outside the elementary power-rule treatment used here.

The Integration Pattern

To integrate

$$x^n,$$

we:

increase the power by 1

and then

divide by the new power.

Thus

$$x^4 \longrightarrow \frac{x^5}{5}.$$

Therefore

The Integration Pattern (continued)

$$\int x^4 dx = \frac{x^5}{5} + C.$$

Example

Evaluate

$$\int 6x^2 dx.$$

Since

$$\int x^2 dx = \frac{x^3}{3},$$

we obtain

$$\int 6x^2 dx = 6 \left(\frac{x^3}{3} \right) + C.$$

Hence

$$\int 6x^2 dx = 2x^3 + C.$$

Example (continued)

Check:

$$\frac{d}{dx}(2x^3 + C) = 6x^2.$$

Integrating a Polynomial

Evaluate

$$\int (3x^2 + 4x - 5) \, dx.$$

Integrate term by term:

$$\int 3x^2 \, dx = x^3,$$

$$\int 4x \, dx = 2x^2,$$

$$\int (-5) \, dx = -5x.$$

Therefore

$$x^3 + 2x^2 - 5x + C.$$

Why Can We Integrate Term by Term?

For suitable functions,

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx.$$

Similarly,

$$\int (f(x) - g(x)) \, dx = \int f(x) \, dx - \int g(x) \, dx.$$

This is why polynomial integration can be performed one term at a time.

Simplify First

Just as in differentiation, algebraic simplification may be the most important first step. Consider

$$\int (x - 2)^2 \, dx.$$

Expand first:

$$(x - 2)^2 = x^2 - 4x + 4.$$

Therefore

$$\int (x - 2)^2 \, dx = \int (x^2 - 4x + 4) \, dx.$$

Complete the Integration

Integrating term by term,

$$\int x^2 dx = \frac{x^3}{3},$$

$$\int (-4x) dx = -2x^2,$$

and

$$\int 4 dx = 4x.$$

Hence

$$\int (x - 2)^2 dx = \frac{x^3}{3} - 2x^2 + 4x + C.$$

TMUA Strategy: Algebra Before Calculus

TMUA Thinking

Before integrating, ask:

Can this expression be simplified?

Can you:

- expand brackets?
- cancel factors?
- rewrite roots as fractional powers?
- rewrite reciprocals as negative powers?

Often the calculus becomes routine once the algebra is exposed.

What Makes an Integral Definite?

Compare

$$\int x^2 dx$$

with

$$\int_a^b x^2 dx.$$

The first is an indefinite integral.
Its answer is a family of functions:

$$\frac{x^3}{3} + C.$$

The second has limits a and b .
Its result is a number.

Evaluating a Definite Integral

Suppose

$$\int_a^b f(x) dx.$$

If

$$F'(x) = f(x),$$

then

$$\boxed{\int_a^b f(x) dx = F(b) - F(a).}$$

This relationship is central to the Fundamental Theorem of Calculus.

Example

Evaluate

$$\int_0^3 x^2 dx.$$

An antiderivative is

$$F(x) = \frac{x^3}{3}.$$

Therefore

$$\int_0^3 x^2 dx = \left[\frac{x^3}{3} \right]_0^3.$$

Thus

$$= \frac{27}{3} - 0 = \boxed{9}.$$

Why Does C Disappear?

Suppose we use

$$F(x) = \frac{x^3}{3} + C.$$

Then

$$F(3) - F(0) = (9 + C) - (0 + C).$$

The constants cancel:

$$9 + C - C = 9.$$

Therefore a definite integral does not require an arbitrary constant in its final value.

A Crucial Distinction

It is tempting to say:

$$\text{integral} = \text{area under a curve.}$$

But this is not always correct.

Geometric area is conventionally non-negative.

A definite integral, however, keeps track of sign.

Common Trap

Regions below the x-axis contribute negatively to a definite integral.

Above the x-Axis

If

$$f(x) \geq 0$$

throughout

$$a \leq x \leq b,$$

then

$$\int_a^b f(x) dx$$

equals the ordinary geometric area between the curve and the x-axis.

In this case,

$$\text{definite integral} = \text{geometric area.}$$

Below the x-Axis

If

$$f(x) \leq 0$$

throughout the interval,
then

$$\int_a^b f(x) dx < 0$$

in the usual non-trivial case.

But the geometric area between the curve and the axis is positive.
Therefore

$$\text{geometric area} = - \int_a^b f(x) dx$$

when the entire region lies below the x-axis.

Example: A Negative Definite Integral

Evaluate

$$\int_{-3}^0 x^3 dx.$$

We obtain

$$\left[\frac{x^4}{4} \right]_{-3}^0.$$

Therefore

$$= 0 - \frac{81}{4}$$

so

$$\int_{-3}^0 x^3 dx = -\frac{81}{4}.$$

But What Is the Geometric Area?

The graph

$$y = x^3$$

lies below the x -axis for

$$-3 < x < 0.$$

Therefore the geometric area is

$$\frac{81}{4}$$

while the definite integral is

$$-\frac{81}{4}.$$

This distinction is fundamental.

A Definite Integral Can Be Zero

Consider

$$\int_{-3}^3 x^5 dx.$$

An antiderivative is

$$\frac{x^6}{6}.$$

Therefore

$$\left[\frac{x^6}{6} \right]_{-3}^3 = \frac{3^6}{6} - \frac{(-3)^6}{6} = 0.$$

Thus

$$\int_{-3}^3 x^5 dx = 0.$$

Does Zero Integral Mean Zero Area?

No.

The graph

$$y = x^5$$

has a region below the x -axis for negative x
and a corresponding region above the x -axis for positive x .

Their signed contributions cancel.

Therefore

$$\text{integral} = 0$$

does not imply

$$\text{geometric area} = 0.$$

Odd Functions and Symmetric Intervals

If f is odd,

$$f(-x) = -f(x),$$

then over a symmetric interval,

$$\int_{-a}^a f(x) dx = 0.$$

The positive and negative contributions cancel.

Examples include

$$x, \quad x^3, \quad x^5.$$

This is often visible without performing any integration.

Even Functions and Symmetry

If

$$f(-x) = f(x),$$

then f is even.

Over a symmetric interval,

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

Examples include

$$x^2, \quad x^4, \quad x^6.$$

Symmetry can dramatically reduce the required calculation.

When a Curve Crosses the x -Axis

Suppose $f(x)$ changes sign inside

$$[a, b].$$

Then

$$\int_a^b f(x) dx$$

combines positive and negative contributions.

But if the question asks for the total geometric area, we must separate the regions at the roots of

$$f(x) = 0.$$

A General Area Strategy

Suppose

$$f(c) = 0$$

for some

$$a < c < b,$$

with $f(x) > 0$ before c and $f(x) < 0$ after c .

Then the geometric area is

$$\int_a^c f(x) dx - \int_c^b f(x) dx.$$

Equivalently,

$$\int_a^b |f(x)| dx.$$

A General Area Strategy (continued)

The first form is usually more directly useful in elementary calculations.

The Fundamental Connection

The Fundamental Theorem of Calculus connects the two major ideas:

differentiation

and

integration.

One important form is

$$\int_a^b f(x) dx = F(b) - F(a)$$

where

$$F'(x) = f(x).$$

What Does $F'(x) = f(x)$ Mean?

The statement

$$F'(x) = f(x)$$

means that F is an antiderivative of f .

So the theorem says:

to evaluate the accumulated signed contribution represented by

$$\int_a^b f(x) dx,$$

find an antiderivative F , then calculate

$$F(b) - F(a).$$

A Second Important Form

Another form relevant here is

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$$

This expression deserves careful thought.
Define

$$G(x) = \int_a^x f(t) dt.$$

Then

$$G'(x) = f(x).$$

Why Use t Inside the Integral?

In

$$G(x) = \int_a^x f(t) dt,$$

the variable t is used inside the integral while x is used as the moving upper limit.
This helps distinguish:

integration variable t

from

upper-limit variable x .

The letter inside the integral is essentially a placeholder.

Accumulation Interpretation

Consider

$$G(x) = \int_a^x f(t) dt.$$

As x moves slightly to the right, a small additional strip is accumulated.
Its contribution is approximately

$$f(x) \Delta x.$$

Thus

$$\frac{\Delta G}{\Delta x} \approx f(x).$$

In the limiting interpretation,

$$G'(x) = f(x).$$

Reversing the Limits

An important property is

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Reversing the direction reverses the sign.
For example,

$$\int_1^4 f(x) dx = 7$$

implies

$$\int_4^1 f(x) dx = -7.$$

Why Does Reversing the Limits Change the Sign?

Using an antiderivative F ,

$$\int_a^b f(x) dx = F(b) - F(a).$$

But

$$\int_b^a f(x) dx = F(a) - F(b).$$

Therefore

$$F(a) - F(b) = -(F(b) - F(a)).$$

Hence

$$\boxed{\int_b^a f(x) dx = - \int_a^b f(x) dx.}$$

Splitting an Integral

A second important property is

$$\boxed{\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.}$$

This expresses the idea that accumulated change over a whole interval can be split into pieces.

Does c Have to Lie Between a and b ?

At first glance, we might expect

$$a < c < b.$$

But algebraically, the identity can still work without that ordering, provided the relevant integrals are well defined.

The orientation of definite integrals automatically handles the signs.

TMUA Thinking

Do not add unnecessary conditions to a mathematical identity.

Example: Recovering a Missing Integral

Suppose

$$\int_2^7 f(x) dx = 12$$

and

$$\int_2^5 f(x) dx = 8.$$

Since

$$\int_2^7 f(x) dx = \int_2^5 f(x) dx + \int_5^7 f(x) dx,$$

we have

$$12 = 8 + \int_5^7 f(x) dx.$$

Example: Recovering a Missing Integral (continued)

Therefore

$$\int_5^7 f(x) dx = 4.$$

Equal Ranges

Suppose two integrals have the same limits.
Then

$$\int_a^b f(x) dx + \int_a^b g(x) dx = \int_a^b (f(x) + g(x)) dx.$$

Similarly,

$$\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b (f(x) - g(x)) dx.$$

Contiguous Ranges

If the integrand is the same and the intervals join, then

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

The middle endpoint b disappears when the intervals are combined.

A TMUA-Style Integral Puzzle

Suppose

$$\int_2^4 f(x) dx = 5$$

and

$$\int_3^4 f(x) dx = 2.$$

Find

$$\int_2^3 f(x) dx.$$

We know

$$\int_2^4 f = \int_2^3 f + \int_3^4 f.$$

A TMUA-Style Integral Puzzle (continued)

Therefore

$$5 = \int_2^3 f(x) dx + 2.$$

Hence

$$\int_2^3 f(x) dx = 3.$$

Be Careful: Not Every Operation Distributes

Linearity allows

$$\int (f + g) dx = \int f dx + \int g dx.$$

But this does **not** mean that arbitrary algebraic operations distribute through integration. For example, in general,

$$\int f(x)g(x) dx \neq \left(\int f(x) dx \right) \left(\int g(x) dx \right).$$

Likewise, one must not invent a “quotient rule” for integrals.

Why Do We Need Approximation?

Sometimes we want an area or definite integral but do not have a convenient exact antiderivative. We can approximate the region using simpler shapes. One important method is the

trapezium rule.

The idea is to replace portions of the curve by straight-line segments and calculate trapezium areas.

Area of One Trapezium

Suppose the parallel vertical sides have lengths

a and b ,

and their horizontal separation is

h .

Then

$$A = \frac{h}{2}(a + b).$$

This simple geometric formula is the foundation of the trapezium rule.

Equal-Width Strips

Suppose we divide an interval into n equal strips of width

$$h.$$

Let the corresponding function values be

$$y_0, y_1, y_2, \dots, y_n.$$

The first trapezium has area

$$\frac{h}{2}(y_0 + y_1).$$

The second has area

$$\frac{h}{2}(y_1 + y_2).$$

And so on.

Deriving the Trapezium Rule

Adding the trapezia,

$$A \approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \dots + \frac{h}{2}(y_{n-1} + y_n).$$

Factor out

$$\frac{h}{2}.$$

Then

$$A \approx \frac{h}{2} [y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n].$$

Why Do Interior Values Appear Twice?

Consider

$$y_1.$$

It is the right side of the first trapezium
and the left side of the second.

Therefore it appears twice.

Similarly,

$$y_2, \dots, y_{n-1}$$

are each shared by two adjacent trapezia.

But

$$y_0$$

and

$$y_n$$

Why Do Interior Values Appear Twice? (continued)

occur only once.

That explains the coefficient pattern

$$1, 2, 2, \dots, 2, 1.$$

Finding the Strip Width

If the interval is

$$[a, b]$$

and we use n equal strips, then

$$h = \frac{b - a}{n}.$$

Be careful:

n is the number of trapezia.

Therefore there are

$$n + 1$$

ordinates/function values.

Trapezium Rule Example

Estimate

$$\int_0^2 x^2 dx$$

using two equal strips.

Here

$$h = \frac{2 - 0}{2} = 1.$$

The x -values are

$$0, 1, 2.$$

Thus

$$y_0 = 0, \quad y_1 = 1, \quad y_2 = 4.$$

Complete the Approximation

Using

$$A \approx \frac{h}{2}(y_0 + 2y_1 + y_2),$$

we get

$$A \approx \frac{1}{2}(0 + 2 + 4).$$

Therefore

$$\boxed{A \approx 3}.$$

The exact integral is

$$\int_0^2 x^2 dx = \frac{8}{3}.$$

So the trapezium-rule estimate is larger.

Why Can the Trapezium Rule Be Wrong?

The trapezium rule replaces the curved graph between adjacent points by a straight chord.

If that chord lies above the curve, the estimate tends to be too large.

If it lies below the curve, the estimate tends to be too small.

Therefore the error depends on the shape of the graph.

A Convex-Upward Example

For

$$y = x^2,$$

a chord joining two points of the graph lies above the curve between those points.
Therefore trapezia built using these chords produce an overestimate of the area.
This explains why

$$3 > \frac{8}{3}$$

in our previous example.

A Concave-Downward Example

For a graph bending the opposite way, the straight chord may lie below the curve.
Then the trapezium-rule estimate may be an underestimate.

TMUA Thinking

Do not merely memorise “trapezium rule overestimates” or “underestimates.”
Look at the geometry of the particular curve.

Sometimes the Answer Is Not Immediately Clear

If the graph changes its bending behaviour over the interval, some trapezia may overestimate while others underestimate.

Therefore it may not always be possible to decide the overall direction of the error by one simple visual rule.

Key Idea

The shape of the graph matters.

Area or Definite Integral?

There is another distinction.

The trapezium rule may be used to estimate

geometric area

or

a definite integral.

These are not automatically the same when the curve lies below the x -axis.
Always read the question carefully.

What Is a Differential Equation Here?

Consider

$$\frac{dy}{dx} = f(x).$$

This tells us the derivative of y .

The problem is to recover y .

So the question is essentially:

What function differentiates to $f(x)$?

This is an integration problem.

Example

Solve

$$\frac{dy}{dx} = 3x^2 + 4x - 3.$$

Integrate both sides with respect to x :

$$y = \int (3x^2 + 4x - 3) dx.$$

Therefore

$y = x^3 + 2x^2 - 3x + C.$

Why Is There Still a Constant?

Knowing

$$\frac{dy}{dx} = 3x^2 + 4x - 3$$

does not uniquely determine y .
All functions

$$y = x^3 + 2x^2 - 3x + C$$

have the same derivative.
So the differential equation alone gives a family of possible solutions.

A Condition Can Determine C

Suppose

$$\frac{dy}{dx} = 3x^2 + 4x - 3$$

and we are also told

$$y = 5 \quad \text{when} \quad x = 1.$$

First integrate:

$$y = x^3 + 2x^2 - 3x + C.$$

Now use the given condition.

Find the Constant

Substitute

$$x = 1, \quad y = 5.$$

Then

$$5 = 1 + 2 - 3 + C.$$

Hence

$$5 = C.$$

Therefore

$$y = x^3 + 2x^2 - 3x + 5.$$

The additional condition has selected one member of the family.

Conceptual Picture

Without an additional condition:

$$\frac{dy}{dx} = f(x)$$

gives

$$y = F(x) + C.$$

With a condition such as

$$y = y_0 \quad \text{when} \quad x = x_0,$$

we can determine C .

Thus

$$\text{derivative information} + \text{one point} \implies \text{specific curve.}$$

Another Way to Use the Condition

Suppose

$$\frac{dy}{dx} = f(x)$$

and

$$y = y_0 \quad \text{when} \quad x = x_0.$$

We can express the accumulated change as

$$y - y_0 = \int_{x_0}^x f(t) dt.$$

This directly connects the derivative with the change in the function value.

Apply It to Our Example

We know

$$\frac{dy}{dx} = 3x^2 + 4x - 3$$

and

$$y = 5 \quad \text{at} \quad x = 1.$$

Therefore

$$y - 5 = \int_1^x (3t^2 + 4t - 3) dt.$$

Integrating,

$$y - 5 = [t^3 + 2t^2 - 3t]_1^x.$$

Complete the Alternative Method

Thus

$$y - 5 = x^3 + 2x^2 - 3x - (1 + 2 - 3).$$

But

$$1 + 2 - 3 = 0.$$

Hence

$$y - 5 = x^3 + 2x^2 - 3x.$$

Therefore

$$y = x^3 + 2x^2 - 3x + 5.$$

Same result, different viewpoint.

TMUA Challenge 1

Suppose

$$F'(x) = f(x)$$

and

$$F(4) = 11, \quad F(1) = 3.$$

Find

$$\int_1^4 f(x) dx.$$

By the Fundamental Theorem,

$$\int_1^4 f(x) dx = F(4) - F(1).$$

Therefore

TMUA Challenge 1 (continued)

$$\boxed{8}.$$

No explicit integration is required.

TMUA Challenge 2

Suppose

$$\int_2^6 f(x) dx = 9.$$

Find

$$\int_6^2 f(x) dx.$$

Reversing the limits changes the sign:

$$\boxed{\int_6^2 f(x) dx = -9.}$$

TMUA Challenge 3

Suppose

$$\int_1^5 f(x) dx = 7$$

and

$$\int_5^8 f(x) dx = -2.$$

Then

$$\int_1^8 f(x) dx = \int_1^5 f(x) dx + \int_5^8 f(x) dx.$$

Therefore

$$\int_1^8 f(x) dx = 5.$$

TMUA Challenge 4

Suppose

$$\int_{-4}^4 f(x) dx = 0.$$

Can we conclude that

$$f(x) = 0$$

throughout the interval?

No.

Positive and negative contributions may cancel.

An odd function over a symmetric interval provides a standard example.

TMUA Challenge 5

Suppose f is odd.

Evaluate

$$\int_{-7}^7 f(x) dx.$$

By symmetry,

$$\boxed{0}.$$

The key step is recognising structure, not finding an antiderivative.

TMUA Challenge 6

Suppose f is even and

$$\int_0^5 f(x) dx = 12.$$

Find

$$\int_{-5}^5 f(x) dx.$$

Since f is even,

$$\int_{-5}^5 f(x) dx = 2 \int_0^5 f(x) dx.$$

Hence

$$\boxed{24}.$$

TMUA Challenge 7

Let

$$G(x) = \int_2^x (t^3 + 1) dt.$$

Find $G'(x)$.

By the Fundamental Theorem of Calculus,

$$G'(x) = x^3 + 1.$$

There is no need to evaluate the integral first.

TMUA Challenge 8

Suppose

$$\frac{dy}{dx} = 6x - 4$$

and

$$y = 7 \quad \text{when} \quad x = 2.$$

Integrating,

$$y = 3x^2 - 4x + C.$$

Using the condition,

$$7 = 12 - 8 + C.$$

Thus

$$C = 3.$$

TMUA Challenge 8 (continued)

Therefore

$$y = 3x^2 - 4x + 3.$$

Common Error 1: Forgetting $+C$

For an indefinite integral,

$$\int 3x^2 dx$$

is not simply

$$x^3.$$

The complete answer is

$$x^3 + C.$$

Without C , you have written only one member of an entire family of antiderivatives.

Common Error 2: Adding C to a Definite Answer

For

$$\int_a^b f(x) dx,$$

the result is a number.

You do not leave an arbitrary $+C$ in the final answer.

Any constant appearing in an antiderivative cancels when calculating

$$F(b) - F(a).$$

Common Error 3: Integral = Area

Be careful with the statement

$$\text{integral} = \text{area}.$$

If the graph lies below the x -axis,
the definite integral contributes negatively,
while geometric area remains positive.

Always determine which quantity the problem actually requests.

Common Error 4: Ignoring Axis Crossings

If the graph crosses the x -axis and the question asks for geometric area, do not blindly calculate one definite integral.

Find the crossing points.

Split the interval.

Then make each geometric contribution positive.

Common Error 5: Reversing Limits Without a Sign

Remember

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Changing the order of the limits changes the sign.

This is not optional notation.

It changes the value.

Common Error 6: Number of Trapezia

If there are n equal trapezia, there are

$$n + 1$$

function values.

For example, four trapezia require five ordinates:

$$y_0, y_1, y_2, y_3, y_4.$$

Do not confuse the number of intervals with the number of sample points.

Common Error 7: Memorising Error Direction

Do not assume the trapezium rule always overestimates.

Do not assume it always underestimates.

Look at how the straight chords compare with the actual curve.

geometry determines the error direction.

Before Calculating Anything

When you see an integral, ask:

- ① Is it definite or indefinite?
- ② Is the question asking for an integral or an area?
- ③ Can symmetry help?
- ④ Can the expression be simplified?
- ⑤ Can integral identities avoid calculation?
- ⑥ Are the intervals contiguous?
- ⑦ Have the limits been reversed?
- ⑧ Does the graph cross the axis?

TMUA Thinking

Do not integrate automatically merely because you see an integral sign.

Three Levels of Integration Thinking

Level 1: Compute

Find an antiderivative.

Level 2: Interpret

Understand signed area and accumulation.

Level 3: Reason

Use symmetry, identities, limits and structure to avoid unnecessary computation.

The TMUA Question Behind the Calculation

A routine question might ask:

$$\int_0^2 x^2 dx.$$

A more reasoning-oriented question may instead give relationships between integrals and ask you to infer another integral.

Or it may ask whether a definite integral is positive, negative or zero.

Or whether a trapezium approximation is too large or too small.

The underlying mathematics is elementary.

The reasoning need not be.

Part 8 Formula Map I

Indefinite integration:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Definite integration:

$$\int_a^b f(x) dx = F(b) - F(a), \quad F'(x) = f(x).$$

Part 8 Formula Map II

Reverse limits:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Split intervals:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Linearity:

$$\int_a^b (f + g) dx = \int_a^b f dx + \int_a^b g dx.$$

Part 8 Formula Map III

Fundamental Theorem:

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$$

Odd symmetry:

$$\int_{-a}^a f(x) dx = 0 \quad \text{if } f \text{ is odd.}$$

Even symmetry:

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f \text{ is even.}$$

Part 8 Formula Map IV

Trapezium rule:

$$A \approx \frac{h}{2} [y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n].$$

For n equal intervals,

$$h = \frac{b - a}{n}.$$

And

$$\text{number of ordinates} = n + 1.$$

Part 8 Formula Map V

Simple differential equation:

$$\frac{dy}{dx} = f(x)$$

gives

$$y = \int f(x) dx = F(x) + C.$$

If

$$y = y_0 \quad \text{when} \quad x = x_0,$$

then the condition determines C .
Alternatively,

$$y - y_0 = \int_{x_0}^x f(t) dt.$$

Part 8 Self-Check

Can You Explain These Without Looking Back?

- 1 Why is integration called reverse differentiation?
- 2 Why does an indefinite integral require $+C$?
- 3 Why does a definite integral not require $+C$?
- 4 What is the difference between signed integral and geometric area?
- 5 Why can a non-zero graph have integral zero?
- 6 What happens when integration limits are reversed?
- 7 How can contiguous integrals be combined?
- 8 What does the Fundamental Theorem connect?
- 9 Why do interior ordinates get coefficient 2 in the trapezium rule?
- 10 How do you decide overestimate versus underestimate?
- 11 How do you solve $\frac{dy}{dx} = f(x)$?
- 12 What role does an additional point condition play?

TMUA Habits for Integration

- 1 Think before integrating.
- 2 Distinguish indefinite from definite integrals.
- 3 Remember the constant of integration.
- 4 Simplify algebraically before applying calculus.
- 5 Distinguish signed integral from geometric area.
- 6 Use symmetry whenever possible.
- 7 Understand, rather than merely memorise, integral identities.
- 8 Recognise the Fundamental Theorem conceptually.
- 9 Derive the trapezium rule from geometry.
- 10 Use graph shape to reason about approximation error.
- 11 Interpret differential equations as reverse differentiation.

The Bigger Mathematical Picture

Differentiation asks:

How is a quantity changing?

Integration asks:

What accumulated change produced this result?

The Fundamental Theorem connects these ideas.

local change $\longleftrightarrow$ global accumulation

This is one of the central ideas of calculus.

Coming in Part 9

TMUA Mathematics Masterclass

Part 9: Graphs of Functions

We will bring together many ideas from the earlier parts:

- recognising standard graph shapes;
- transformations of graphs;
- translations and scalings;
- reflections;
- roots and intercepts;
- polynomial behaviour;
- using differentiation to determine graph shape;
- stationary points;
- increasing and decreasing behaviour;

- intersections of graphs;
- graphical interpretation of simultaneous equations;
- TMUA graph reasoning.

Integration Is More Than Finding an Area.

It connects change, accumulation, geometry and functions.

$$F'(x) = f(x) \iff \int f(x) dx = F(x) + C$$

and

$$\int_a^b f(x) dx = F(b) - F(a)$$

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TMUA Mathematics Masterclass

Part 9: Graphs of Functions

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Why Graphs Matter

An equation gives us an algebraic description.

A graph gives us a geometric description.

For example,

$$y = x^2 - 4x + 3$$

contains information about:

- roots,
- intercepts,
- turning points,
- increasing and decreasing behaviour,
- symmetry,
- and possible solutions to equations.

Key Idea

A graph is not merely a picture of an equation.

It is another mathematical representation of the same relationship.

The Algebra–Geometry Connection

Consider

$$y = x + 2.$$

Every point on this line corresponds to an ordered pair

$$(x, y)$$

satisfying

$$y = x + 2.$$

For example,

$$(1, 3), \quad (2, 4), \quad (-2, 0)$$

all lie on the graph.

So we can think of the graph as

The Algebra–Geometry Connection (continued)

a geometric representation of all solutions.

Part 9 Roadmap

We will study:

- ➊ Recognising common graph families
- ➋ Polynomial shapes
- ➌ Leading coefficients and end behaviour
- ➍ Modulus graphs
- ➎ Modulus inequalities
- ➏ Vertical stretches and reflections
- ➐ Vertical translations
- ➑ Horizontal translations
- ➒ Horizontal scaling
- ➓ Combined transformations
- ➑ Composite functions
- ➒ The family $y = mx + c$
- ➓ The family $y = a(x + b)^2 + c$

Part 9 Roadmap (continued)

- 14 Differentiation and graph shape
- 15 Roots and intercepts
- 16 Intersections and simultaneous equations
- 17 TMUA graph reasoning

Graphs You Should Recognise

You should be comfortable recognising and sketching the main families appearing throughout the mathematics specification:

Lines

Quadratics

Cubics

Trigonometric functions

Exponentials and logarithms

Square-root functions

Graphs You Should Recognise (continued)

Modulus functions

Recognition should become increasingly automatic.

Linear Functions

The general linear function is

$$y = mx + c.$$

Its graph is a straight line.

The parameter m determines the gradient.

The parameter c determines the y -intercept.

For example,

$$y = 2x + 3$$

has gradient

$$2$$

and crosses the y -axis at

$$(0, 3).$$

Quadratic Functions

The basic quadratic

$$y = x^2$$

has a U-shaped graph.
More generally,

$$y = ax^2 + bx + c, \quad a \neq 0.$$

If

$$a > 0,$$

the parabola opens upward.
If

$$a < 0,$$

the parabola opens downward.

Quadratic Functions (continued)

Key Idea

The sign of the highest-power coefficient strongly influences the overall shape.

Cubic Functions

The simplest cubic is

$$y = x^3.$$

A general cubic has the form

$$y = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

Its detailed shape can vary.

But its end behaviour is strongly influenced by the sign of a .

For $a > 0$,

$$x \rightarrow \infty \implies y \rightarrow \infty,$$

while

$$x \rightarrow -\infty \implies y \rightarrow -\infty.$$

Higher-Degree Polynomials

You should also develop intuition for graphs such as

$$x^4, \quad x^5,$$

and more general quartics and quintics.

The exact shape depends on all coefficients.

However, the highest-degree term becomes especially important for large values of $|x|$.

For example,

$$x^4 - 100x^2 + 7$$

eventually behaves like

$$x^4.$$

Even Degree vs Odd Degree

For a polynomial whose leading term is

$$ax^n,$$

the parity of n helps determine the end behaviour.

If n is even, the two ends ultimately move in the same vertical direction.

If n is odd, the two ends ultimately move in opposite directions.

The sign of a determines which direction.

Four End-Behaviour Patterns

For positive leading coefficient:

even degree: both ends upward

odd degree: left down, right up

For negative leading coefficient:

even degree: both ends downward

odd degree: left up, right down

This gives useful information before any detailed calculation.

Exponential Functions

A typical exponential function is

$$y = a^x, \quad a > 0, \quad a \neq 1.$$

For

$$a > 1,$$

the function increases rapidly.
Also,

$$a^0 = 1,$$

so the graph passes through

$$(0, 1).$$

The graph remains above the x -axis.

Logarithmic Functions

The logarithmic function

$$y = \log_a x$$

is connected to

$$y = a^x$$

through inverse-function ideas.
Its domain requires

$$x > 0.$$

Since

$$\log_a 1 = 0,$$

the graph passes through

Logarithmic Functions (continued)

$$(1, 0).$$

Square-Root Function

The basic square-root graph is

$$y = \sqrt{x}.$$

For real-valued y ,

$$x \geq 0.$$

So the graph begins at

$$(0, 0)$$

and extends to the right.

The domain restriction is part of the graph.

TMUA Thinking

Always ask where the function is actually defined.

Trigonometric Graphs

You should recognise standard graphs such as

$$y = \sin x$$

and

$$y = \cos x.$$

Important features include:

- periodicity,
- amplitude,
- roots,
- maxima and minima,
- transformations.

Graph transformations become particularly useful for trigonometric functions.

What Does Modulus Mean?

For a real number u ,

$$|u|$$

represents its non-negative magnitude.
Equivalently,

$$|u| = \begin{cases} u, & u \geq 0, \\ -u, & u < 0. \end{cases}$$

For example,

$$|5| = 5,$$

and

$$|-5| = 5.$$

A Geometric Interpretation of Modulus

Consider

$$|x - 3|.$$

This can be interpreted as

the distance of x from 3

on the number line.

Therefore

$$|x - 3| = 2$$

means that x lies two units from 3.

Hence

$$x = 1 \quad \text{or} \quad x = 5.$$

Graph of $y = |x - 3|$

We can write

$$|x - 3| = \begin{cases} x - 3, & x \geq 3, \\ 3 - x, & x < 3. \end{cases}$$

So the graph consists of two straight-line pieces meeting at

$$(3, 0).$$

The resulting graph has a V-shape.

From $y = f(x)$ to $y = |f(x)|$

Suppose we already know the graph of

$$y = f(x).$$

Then

$$y = |f(x)|$$

keeps all portions with

$$f(x) \geq 0$$

unchanged.

Any portion for which

$$f(x) < 0$$

is reflected in the x -axis.

Thus the transformed graph never lies below the x -axis.

Piecewise Interpretation

We may write

$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0, \\ -f(x), & f(x) < 0. \end{cases}$$

Therefore the “flipped” pieces have equation

$$y = -f(x),$$

while the unchanged pieces retain

$$y = f(x).$$

This is useful when modulus equations or inequalities must be solved algebraically and graphically.

A Different Transformation: $f(|x|)$

Do not confuse

$$|f(x)|$$

with

$$f(|x|).$$

For

$$y = f(|x|),$$

positive and negative inputs of the same magnitude produce the same output.

Indeed,

$$f(|-x|) = f(|x|).$$

So the resulting graph is symmetric about the y -axis.

How to Construct $y = f(|x|)$

Start with

$$y = f(x).$$

Keep the portion for

$$x \geq 0.$$

Then reflect that right-hand portion in the y -axis.

This constructs

$$y = f(|x|).$$

Common Trap

$y = |f(x)|$ involves reflection in the x -axis.

$y = f(|x|)$ involves symmetry about the y -axis.

A Graphical Modulus Inequality

Consider

$$x + 4 < |2x - 1|.$$

Sketch

$$y = x + 4$$

and

$$y = |2x - 1|.$$

The inequality asks:

When is $y = x + 4$ below $y = |2x - 1|$?

So first find the intersection points.

Finding the Intersections

On one branch,

$$x + 4 = 2x - 1.$$

Hence

$$x = 5.$$

On the reflected branch,

$$x + 4 = -(2x - 1).$$

Therefore

$$x + 4 = -2x + 1,$$

so

$$3x = -3$$

Finding the Intersections (continued)

and

$$x = -1.$$

Complete the Inequality

The two graphs intersect at

$$x = -1 \quad \text{and} \quad x = 5.$$

From the graph, the straight line lies below the modulus graph outside these intersections.
Therefore

$$x < -1 \quad \text{or} \quad x > 5.$$

This is a good example of algebra and graph reasoning working together.

Four Core Transformations

Starting from

$$y = f(x),$$

we need to understand

$$y = af(x)$$

$$y = f(x) + a$$

$$y = f(x + a)$$

and

$$y = f(ax).$$

The aim is not merely to memorise these transformations.

We want to understand why they work.

Transformation 1: $y = af(x)$

Suppose

$$y = f(x).$$

Then

$$y = af(x)$$

multiplies every output by a .

A point

$$(x, y)$$

on the original graph therefore becomes

$$(x, ay).$$

Only the vertical coordinate changes.

Example: $y = 4f(x)$

Suppose

$$f(x) = x^3.$$

Then

$$4f(x) = 4x^3.$$

At every x ,
the new y -coordinate is four times the old y -coordinate.
Therefore the graph undergoes a vertical stretch away from the x -axis.

vertical scale factor 4

What If $0 < a < 1$?

Suppose

$$y = \frac{1}{2}f(x).$$

Then every vertical coordinate is halved.
So the graph becomes less tall:

$$(x, y) \longrightarrow \left(x, \frac{y}{2}\right).$$

The graph is compressed vertically toward the x -axis.

What If $a < 0$?

Consider

$$y = -2f(x).$$

The magnitude 2 changes the vertical scale.

The minus sign changes

$$y \longrightarrow -y.$$

Therefore there is also a reflection in the x-axis.

So

$$y = af(x), \quad a < 0$$

combines vertical scaling with reflection in the x-axis.

Transformation 2: $y = f(x) + a$

Here the value a is added to every output.

Thus

$$(x, y)$$

becomes

$$(x, y + a).$$

Therefore

$$y = f(x) + a$$

is a vertical translation.

The translation vector is

$$\begin{pmatrix} 0 \\ a \end{pmatrix}.$$

Example

Start with

$$y = x^2.$$

Now consider

$$y = x^2 + 3.$$

Every point moves upward by three units.
The vertex

$$(0, 0)$$

becomes

$$(0, 3).$$

Therefore the entire parabola is translated upward by 3.

If $a < 0$

Consider

$$y = f(x) - 4.$$

This is

$$y = f(x) + (-4).$$

Therefore the graph moves downward by 4.
Its translation vector is

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

Transformation 3: $y = f(x + a)$

This is one of the most common sources of errors.
At first sight,

$$f(x + a)$$

may appear as though the graph should move right when $a > 0$.
But it moves in the opposite direction.

$$y = f(x + a)$$

is a translation by

$$\begin{pmatrix} -a \\ 0 \end{pmatrix}.$$

Why Does the Direction Reverse?

Suppose a feature of $f(x)$ occurs when

$$x = r.$$

For the transformed function

$$f(x + a),$$

we need

$$x + a = r.$$

Therefore

$$x = r - a.$$

So the same feature now occurs a units to the left.
That explains the apparently reversed direction.

Example: $f(x) = x^2$

The original graph

$$y = x^2$$

has its vertex at

$$x = 0.$$

Now consider

$$y = (x + 2)^2.$$

Its vertex occurs when

$$x + 2 = 0.$$

Hence

$$x = -2.$$

So the graph has moved two units to the left.

Example: $f(x - 3)$

If

$$f(x) = x^2,$$

then

$$f(x - 3) = (x - 3)^2.$$

The vertex occurs when

$$x - 3 = 0,$$

so

$$x = 3.$$

Thus

$$\boxed{f(x - 3)}$$

moves the graph three units to the right.

Substitution Must Affect Every x

Suppose

$$f(x) = x^2 + 2x - 5.$$

Find $f(x + 3)$.

Replace every x by $x + 3$:

$$f(x + 3) = (x + 3)^2 + 2(x + 3) - 5.$$

Do not replace only some occurrences of x .

Common Trap

Function substitution acts on the entire input expression.

Transformation 4: $y = f(ax)$

Now consider

$$y = f(ax).$$

The transformation happens inside the input.

If a feature originally occurs at

$$x = r,$$

then in the transformed graph we need

$$ax = r.$$

Thus

$$x = \frac{r}{a}.$$

So horizontal coordinates are divided by a .

Example: $f(2x)$

Suppose a feature of $f(x)$ occurs at

$$x = 6.$$

In

$$f(2x),$$

the corresponding feature satisfies

$$2x = 6.$$

Therefore

$$x = 3.$$

The feature moves from 6 to 3.

Hence the graph is compressed horizontally toward the y -axis.

Coordinate Mapping

For

$$y = f(ax),$$

a point

$$(r, f(r))$$

on the original graph appears at

$$\left(\frac{r}{a}, f(r)\right)$$

on the transformed graph.

Thus the y -coordinate remains unchanged,
while the x -coordinate is divided by a .

What If $0 < a < 1$?

Suppose

$$y = f\left(\frac{x}{2}\right).$$

A feature originally at

$$x = r$$

now satisfies

$$\frac{x}{2} = r.$$

Hence

$$x = 2r.$$

So the graph expands horizontally away from the y -axis.

What If $a < 0$?

For

$$y = f(-x),$$

we have

$$x \longrightarrow -x.$$

Therefore the graph is reflected in the y -axis.

More generally,

$$y = f(ax), \quad a < 0,$$

involves horizontal scaling together with reflection in the y -axis.

The Four Transformations

Function	Effect
$af(x)$	vertical scaling; reflect in x -axis if $a < 0$
$f(x) + a$	translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$f(x + a)$	translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$f(ax)$	horizontal scaling; reflect in y -axis if $a < 0$

TMUA Thinking

Outside changes affect outputs.
Inside changes affect inputs.

A Useful Memory Principle

Compare

$$af(x)$$

with

$$f(ax).$$

In

$$af(x),$$

the multiplication occurs after the function produces its output.

So it changes vertical coordinates.

In

$$f(ax),$$

the multiplication occurs before the function acts.

So it changes horizontal coordinates.

Transformations Can Be Combined

Expressions often contain more than one transformation.
For example,

$$y = 3f(2x - 4) + 5.$$

This involves:

- horizontal change,
- vertical scaling,
- vertical translation.

But combined transformations require care.

Common Trap

The order and algebraic structure matter.

A Trigonometric Example

Consider

$$f(x) = \cos x$$

and

$$f\left(2x + \frac{\pi}{6}\right).$$

We want to understand

$$\cos\left(2x + \frac{\pi}{6}\right).$$

Factor the input:

$$2x + \frac{\pi}{6} = 2\left(x + \frac{\pi}{12}\right).$$

Therefore

A Trigonometric Example (continued)

$$\cos\left(2x + \frac{\pi}{6}\right) = \cos\left(2\left(x + \frac{\pi}{12}\right)\right).$$

Read the Structure Carefully

The expression

$$f\left(2\left(x + \frac{\pi}{12}\right)\right)$$

helps us see the horizontal structure.

The graph is horizontally compressed according to the factor 2, with the corresponding horizontal displacement determined by

$$x + \frac{\pi}{12}.$$

The important lesson is broader than this example:

rewrite the input before interpreting the graph.

Why Transformation Order Can Matter

Suppose you first create

$$f(2x)$$

and then replace x by

$$x + \frac{\pi}{6}.$$

You obtain

$$f\left(2\left(x + \frac{\pi}{6}\right)\right) = f\left(2x + \frac{\pi}{3}\right),$$

which is not

$$f\left(2x + \frac{\pi}{6}\right).$$

So careless sequencing changes the function itself.

The Notation $f(g(x))$

The expression

$$f(g(x))$$

is read as

$$\boxed{f \text{ of } g \text{ of } x}.$$

It means:

- ① start with x ;
- ② apply g ;
- ③ use the resulting output as the input to f .

Symbolically,

$$x \longrightarrow g(x) \longrightarrow f(g(x)).$$

Example

Let

$$g(x) = 2x$$

and

$$f(x) = x^2 + 3x - 2.$$

Then

$$f(g(x)) = [g(x)]^2 + 3g(x) - 2.$$

Substitute

$$g(x) = 2x.$$

Thus

$$f(g(x)) = (2x)^2 + 3(2x) - 2.$$

Example (continued)

Therefore

$$f(g(x)) = 4x^2 + 6x - 2.$$

Composition Is Usually Order-Sensitive

In general,

$$f(g(x)) \neq g(f(x)).$$

For example, let

$$f(x) = x^2$$

and

$$g(x) = x + 1.$$

Then

$$f(g(x)) = (x + 1)^2,$$

while

$$g(f(x)) = x^2 + 1.$$

These are different functions.

Function Machines

It can help to imagine functions as machines.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x)).$$

Changing the order gives

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)).$$

Different order means a different chain of operations.
Therefore different outputs should usually be expected.

Understanding $y = mx + c$

Consider

$$y = mx + c.$$

The parameter m controls the gradient.

The parameter c controls the vertical intercept.

We can understand the family through transformations starting from

$$y = x.$$

Changing c

Start with

$$y = mx.$$

Then

$$y = mx + c$$

adds c to every output.

Therefore changing c translates the line vertically.

The gradient remains m .

So all lines

$$y = mx + c$$

with the same m are parallel.

Changing m

The coefficient m controls the gradient.
For

$$m > 0,$$

the line rises as x increases.
For

$$m < 0,$$

the line falls as x increases.
For

$$m = 0,$$

the function becomes

$$y = c,$$

a horizontal line.

Intercepts of $y = mx + c$

The y -intercept occurs when

$$x = 0.$$

Therefore

$$y = c.$$

So the point is

$$(0, c).$$

If $m \neq 0$, the x -intercept occurs when

$$0 = mx + c.$$

Hence

$$x = -\frac{c}{m}.$$

The Form $y = a(x + b)^2 + c$

Consider

$$y = a(x + b)^2 + c.$$

This form makes the transformations of

$$y = x^2$$

particularly visible.

The parameters a, b, c each play a distinct role.

What Does b Do?

Consider

$$(x + b)^2.$$

This is the transformation

$$f(x + b)$$

where

$$f(x) = x^2.$$

Therefore the parabola moves horizontally by

$$-b.$$

Its axis of symmetry is

$$x = -b.$$

What Does c Do?

The term

$$+c$$

moves every output vertically by c .

Therefore the vertex has vertical coordinate

$$c.$$

Combining this with the horizontal translation gives the vertex

$$\boxed{(-b, c)}.$$

What Does a Do?

The coefficient a changes the vertical scale.

For

$$a > 0,$$

the parabola opens upward.

For

$$a < 0,$$

it opens downward.

The magnitude

$$|a|$$

controls the vertical scaling.

Thus a controls both orientation and shape.

Reading a Quadratic Immediately

Consider

$$y = -3(x - 2)^2 + 5.$$

Write

$$x - 2 = x + (-2).$$

Therefore

$$b = -2.$$

Hence the vertex is

$$(2, 5).$$

Since

$$a = -3 < 0,$$

Reading a Quadratic Immediately (continued)

the parabola opens downward.

The factor 3 changes its vertical scale.

A Common Sign Trap

For

$$y = a(x + b)^2 + c,$$

the vertex is

$$\boxed{(-b, c)},$$

not

$$(b, c).$$

Why?

Because the squared term is zero when

$$x + b = 0.$$

Therefore

A Common Sign Trap (continued)

$$x = -b.$$

Understanding this is better than memorising a sign rule.

Using Differentiation to Understand Graphs

Differentiation provides information about the shape of a function.

If

$$f'(x) > 0,$$

then the function is strictly increasing on the relevant interval.

If

$$f'(x) < 0,$$

then the function is strictly decreasing.

Stationary points occur where

$$f'(x) = 0.$$

Stationary Points

A stationary point occurs where the tangent has zero gradient.

Therefore

$$f'(x) = 0.$$

The stationary point may be:

- a local maximum;
- a local minimum;
- or, more generally, another stationary behaviour.

For the TMUA material here, maxima and minima are the central differentiation cases.

Maximum or Minimum?

Suppose

$$f'(r) = 0.$$

If the derivative changes

$$- \longrightarrow +$$

through r ,
then f changes from decreasing to increasing.
So r corresponds to a local minimum.
If

$$+ \longrightarrow -$$

then f changes from increasing to decreasing.
So r corresponds to a local maximum.

Second Derivative Information

At a stationary point,

$$f'(r) = 0.$$

If

$$f''(r) > 0,$$

this is sufficient for a local minimum in the standard smooth setting considered here.
If

$$f''(r) < 0,$$

this is sufficient for a local maximum.

Common Trap

If $f''(r) = 0$, this test alone does not settle the question.

Why $f''(r) = 0$ Is Inconclusive

Consider

$$f(x) = x^4.$$

At

$$x = 0,$$

we have

$$f'(0) = 0$$

and

$$f''(0) = 0.$$

Yet $x = 0$ is a minimum.
Similarly,

Why $f''(r) = 0$ Is Inconclusive (continued)

$$f(x) = -x^4$$

has a maximum at $x = 0$, even though the second derivative there is also zero.

Finding the x -Intercepts

The graph

$$y = f(x)$$

intersects the x -axis where

$$y = 0.$$

Therefore we solve

$$f(x) = 0.$$

Thus:

$$x\text{-intercepts} \longleftrightarrow \text{real roots of } f(x) = 0.$$

Finding the y -Intercept

The graph intersects the y -axis where

$$x = 0.$$

Therefore the y -coordinate is

$$f(0).$$

Hence the y -intercept is

$$(0, f(0)).$$

This is often one of the quickest pieces of information to obtain when sketching a graph.

Example

Consider

$$f(x) = x^2 - 5x + 6.$$

Factorise:

$$f(x) = (x - 2)(x - 3).$$

Therefore the roots are

$$x = 2, \quad x = 3.$$

So the graph crosses the x -axis at

$$(2, 0)$$

and

$$(3, 0).$$

Quadratic Roots and the Graph

For

$$ax^2 + bx + c = 0,$$

the discriminant is

$$\Delta = b^2 - 4ac.$$

If

$$\Delta > 0,$$

there are two distinct real roots.

Graphically, the parabola intersects the x -axis at two distinct points.

Repeated Root

If

$$\Delta = 0,$$

there is one repeated real root.

Graphically, the parabola touches the x -axis at one point.

Thus algebraically,

$$\Delta = 0$$

corresponds geometrically to tangency to the x -axis.

No Real Roots

If

$$\Delta < 0,$$

the quadratic has no real roots.

Therefore the corresponding parabola does not intersect the x -axis.

So the discriminant is not merely an algebraic device.

It describes the geometry of the graph.

General Polynomial Roots

For a polynomial

$$P(x),$$

every real solution of

$$P(x) = 0$$

corresponds to an x -axis intersection.

Thus understanding the possible number of real roots is equivalent to thinking about how many times the graph can meet the x -axis.

TMUA Thinking

Translate between “root” and “intersection” automatically.

Where Do Two Graphs Intersect?

Suppose we have

$$y = f(x)$$

and

$$y = g(x).$$

At an intersection point, both equations have the same x and y .
Therefore

$$f(x) = g(x).$$

Solving this equation gives the x -coordinates of the intersection points.

Graphical Meaning of Simultaneous Equations

Suppose

$$y = f(x)$$

and

$$y = g(x).$$

A simultaneous solution must satisfy both equations.

Graphically, this means the point must belong to both graphs.

Therefore:

simultaneous solutions = intersection points.

Line and Quadratic

Consider a straight line and a parabola.

There are three standard possibilities:

- ① the line crosses the parabola twice;
- ② the line is tangent to the parabola;
- ③ the line does not meet the parabola in real points.

These correspond algebraically to:

$$2, 1, 0$$

distinct real solutions.

Example

Solve simultaneously

$$y = x^2 + 3x + 2$$

and

$$y = x + 1.$$

At an intersection,

$$x^2 + 3x + 2 = x + 1.$$

Therefore

$$x^2 + 2x + 1 = 0.$$

Hence

$$(x + 1)^2 = 0.$$

Example (continued)

So

$$x = -1.$$

Interpret the Result

When

$$x = -1,$$

the line gives

$$y = 0.$$

So the only intersection is

$$\boxed{(-1, 0)}.$$

Because the resulting quadratic has a repeated root, the line touches the parabola rather than crossing it.

Therefore it is tangent to the parabola at this point.

Solving $f(x) = k$

The equation

$$f(x) = k$$

can be interpreted as finding intersections between

$$y = f(x)$$

and the horizontal line

$$y = k.$$

Therefore the number of solutions equals the number of intersection points.

This can often be determined from a sketch without solving explicitly.

Solving $f(x) = g(x)$

Likewise,

$$f(x) = g(x)$$

asks:

Where do the graphs of f and g meet?

This viewpoint is especially useful when exact algebra is awkward but the possible number or location of solutions can be deduced graphically.

Inequalities as Graph Problems

The inequality

$$f(x) > g(x)$$

asks:

When does the graph of f lie above the graph of g ?

Similarly,

$$f(x) < g(x)$$

asks when the graph of f lies below the graph of g .

This viewpoint connects directly with the modulus inequality examples.

Before Sketching a Function

Ask:

- ① What family does the function belong to?
- ② What is its domain?
- ③ What are its intercepts?
- ④ What is its leading-term behaviour?
- ⑤ Is there symmetry?
- ⑥ Is it a transformation of a standard graph?
- ⑦ Are there stationary points?
- ⑧ Where is it increasing or decreasing?
- ⑨ What does the graph do for large $|x|$?

You rarely need random point plotting.

Structural Graph Sketching

For a polynomial, a useful approach is:

degree



leading coefficient



roots



Structural Graph Sketching (continued)

stationary behaviour



overall shape

The graph should emerge from structure.

Do Not Over-Calculate

A TMUA problem may not require an exact graph.

You may only need to know:

- how many roots exist;
- whether an intersection exists;
- whether a function is positive;
- which graph lies above another;
- how a parameter changes the graph;
- or which sketch is possible.

TMUA Thinking

Extract only the information the problem actually needs.

TMUA Challenge 1

The graph of

$$y = f(x)$$

contains the point

$$(5, 7).$$

Which point must lie on

$$y = f(x + 2)?$$

We need

$$x + 2 = 5.$$

Therefore

$$x = 3.$$

TMUA Challenge 1 (continued)

Hence the transformed graph contains

$$\boxed{(3, 7)}.$$

TMUA Challenge 2

The graph of $y = f(x)$ contains

$$(6, -4).$$

Which point lies on

$$y = f(3x)?$$

We require

$$3x = 6.$$

Hence

$$x = 2.$$

The y -coordinate remains unchanged.
Therefore

TMUA Challenge 2 (continued)

$$(2, -4).$$

TMUA Challenge 3

Suppose $f(x)$ has roots

$$-2, \quad 1, \quad 5.$$

What are the roots of

$$f(x - 3)?$$

We require

$$x - 3 = -2, \quad 1, \quad 5.$$

Thus

$$x = 1, \quad 4, \quad 8.$$

Therefore the new roots are

$$\boxed{1, 4, 8}.$$

TMUA Challenge 4

Suppose

$$f(x) = x^2 - 4.$$

How many solutions does

$$|f(x)| = 3$$

have?

We solve

$$|x^2 - 4| = 3.$$

Therefore

$$x^2 - 4 = 3$$

or

TMUA Challenge 4 (continued)

$$x^2 - 4 = -3.$$

Hence

$$x^2 = 7$$

or

$$x^2 = 1.$$

There are

4

real solutions.

TMUA Challenge 5

Consider

$$y = a(x + 4)^2 - 3.$$

Without expanding, identify the vertex.

The squared term is zero when

$$x = -4.$$

Therefore the vertex is

$(-4, -3)$.

If $a > 0$, it is a minimum.

If $a < 0$, it is a maximum.

TMUA Challenge 6

Suppose

$$f'(x) > 0$$

for every x in

$$1 < x < 5.$$

What can we conclude?

The function is strictly increasing on this interval.

Therefore if

$$1 < x_1 < x_2 < 5,$$

then

$$f(x_1) < f(x_2).$$

TMUA Challenge 7

Suppose the equation

$$f(x) = 3$$

has exactly four real solutions.

What does this mean graphically?

The horizontal line

$$y = 3$$

intersects

$$y = f(x)$$

at exactly four points.

No explicit formula for f is required to interpret the statement.

TMUA Challenge 8

Suppose a line

$$y = mx + c$$

is tangent to a quadratic

$$y = ax^2 + bx + d.$$

Then

$$ax^2 + bx + d = mx + c$$

must have exactly one repeated real solution.

Therefore the discriminant of the resulting quadratic must satisfy

$$\Delta = 0.$$

This converts geometry into algebra.

Common Error 1: Horizontal Translation

Do not say

$$f(x + 3)$$

moves the graph right by 3.

It moves it left by 3.

$$f(x + a) \longrightarrow \text{translation by } \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

Understand this from the input equation rather than memorising an arbitrary rule.

Common Error 2: $f(ax)$

Do not interpret

$$f(2x)$$

as a horizontal stretch by 2.

Features originally at $x = r$ now occur where

$$2x = r.$$

Therefore

$$x = \frac{r}{2}.$$

The graph is compressed horizontally toward the y -axis.

Common Error 3: $af(x)$ vs $f(ax)$

These are very different:

$$2f(x)$$

changes outputs.

But

$$f(2x)$$

changes inputs.

Thus

$$2f(x) \neq f(2x)$$

in general.

Common Error 4: Modulus Transformations

Do not confuse

$$|f(x)|$$

and

$$f(|x|).$$

For

$$|f(x)|,$$

negative output portions are reflected above the x -axis.

For

$$f(|x|),$$

the positive- x portion is mirrored across the y -axis.

Common Error 5: Composition

In general,

$$f(g(x)) \neq g(f(x)).$$

Composition has an order.

Always identify:

which function acts first?

In

$$f(g(x)),$$

g acts first.

Common Error 6: Vertex Sign

For

$$y = a(x + b)^2 + c,$$

the vertex is

$$(-b, c).$$

The horizontal coordinate is not b .

Set

$$x + b = 0$$

to understand why.

Common Error 7: Roots vs Turning Points

A root satisfies

$$f(x) = 0.$$

A stationary point satisfies

$$f'(x) = 0.$$

These are completely different conditions.

A point may be both in a special case, but one does not generally imply the other.

Transformation Map

$$af(x) : \text{vertical scaling}$$

$$f(x) + a : \text{vertical translation}$$

$$f(x + a) : \text{horizontal translation in opposite sign}$$

$$f(ax) : \text{reciprocal horizontal scaling}$$

Negative multipliers additionally produce reflections in the corresponding axis.

Graph–Algebra Map

$$f(x) = 0 \iff \text{x-axis intersections}$$

$$f(x) = g(x) \iff \text{intersections of } y = f(x) \text{ and } y = g(x)$$

$$f(x) > g(x) \iff f \text{ lies above } g$$

$$f'(x) > 0 \implies f \text{ strictly increasing}$$

$$f'(x) < 0 \implies f \text{ strictly decreasing}$$

Quadratic Graph Map

For

$$y = a(x + b)^2 + c,$$

we have

$$\text{vertex} = (-b, c)$$

and

$$a > 0 : \text{opens upward}$$

$$a < 0 : \text{opens downward.}$$

The magnitude $|a|$ determines the vertical scaling.

Discriminant Map

For

$$ax^2 + bx + c = 0,$$

$$b^2 - 4ac > 0 \implies 2 \text{ distinct real roots}$$

$$b^2 - 4ac = 0 \implies 1 \text{ repeated real root}$$

$$b^2 - 4ac < 0 \implies 0 \text{ real roots.}$$

Graphically these correspond to two intersections, tangency, and no x-axis intersection.

Part 9 Self-Check

Can You Explain These Without Looking Back?

- 1 What determines the end behaviour of a polynomial?
- 2 How do you construct $y = |f(x)|$?
- 3 How is $f(|x|)$ different?
- 4 Why does $f(x + a)$ move opposite to the sign of a ?
- 5 What happens to horizontal coordinates under $f(ax)$?
- 6 What happens when $a < 0$ in $af(x)$?
- 7 Why is composition order important?
- 8 What do m and c do in $mx + c$?
- 9 What do a, b, c do in $a(x + b)^2 + c$?
- 10 How does differentiation reveal graph shape?
- 11 What is the graphical meaning of $f(x) = g(x)$?
- 12 What does a repeated quadratic root mean geometrically?

TMUA Habits for Graph Problems

- 1 Recognise the graph family first.
- 2 Use structure instead of random point plotting.
- 3 Check the domain.
- 4 Find intercepts strategically.
- 5 Use leading terms for polynomial end behaviour.
- 6 Understand transformations rather than memorising them.
- 7 Treat inside and outside transformations differently.
- 8 Use differentiation only when it adds useful information.
- 9 Translate equations into intersections.
- 10 Translate inequalities into relative graph position.
- 11 Use symmetry whenever possible.
- 12 Sketch only as accurately as the reasoning requires.

The Central Lesson of Part 9

A strong graph thinker can move fluently between

equation



function



graph



The Central Lesson of Part 9 (continued)

geometric behaviour.

TMUA questions often reward precisely this flexibility.

From Formula Recognition to Graph Intelligence

When you see

$$y = -2(x - 3)^2 + 5,$$

you should not need to expand it immediately.
You should already see:

downward-opening parabola

with vertex

$$(3, 5).$$

Likewise, when you see

$$f(x + 4),$$

you should immediately anticipate a horizontal shift.
The goal is recognition before calculation.

The Bigger Mathematical Picture

Graphs connect nearly everything we have studied:

Algebra

through equations and roots;

Functions

through inputs and outputs;

Trigonometry

through periodic curves;

Exponentials and Logarithms

through inverse relationships;

Calculus

through gradients, stationary points and accumulated change.

End of the Mathematics 2 Content Sequence

TMUA Mathematics Masterclass

Parts 1–9 have now developed the main mathematical content sequence from the supplied Mathematics notes.

Knowledge → Connections → Reasoning → Performance

Do Not Merely Draw Graphs.

Learn to Read Mathematics Through Them.

Algebra $\longleftrightarrow$ Geometry

Equation $\longleftrightarrow$ Graph

Calculation $\longleftrightarrow$ Interpretation

TMUA Mathematics Masterclass

Part 10: TMUA Problem Solving, Strategy and Mathematical Maturity

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You Know the Mathematics. What Comes Next?

Parts 1–9 developed the mathematical toolkit.

But university admissions mathematics requires another layer:

knowing mathematics

is not identical to

solving unfamiliar mathematics efficiently.

A student may know every formula in the syllabus and still struggle with TMUA questions.

Why?

Because TMUA tests how mathematical knowledge is **used**.

The Four Layers of Preparation

A strong preparation programme should develop four layers:

Layer 1: Mathematical Knowledge

Layer 2: Problem-Solving Ability

Layer 3: Mathematical Reasoning

Layer 4: Examination Performance

Weakness in any one layer can limit the final performance.

The Preparation Equation

A useful model is

$$\text{Performance} = \text{Knowledge} \times \text{Reasoning} \times \text{Problem Solving} \times \text{Execution}$$

This is not a literal numerical formula.

It expresses an important principle:

one serious weakness can limit the whole system.

Part 10 Roadmap

We will discuss:

- ① Understanding TMUA Paper 1
- ② Understanding TMUA Paper 2
- ③ The TMUA problem-solving mindset
- ④ How to attack unfamiliar problems
- ⑤ Past-paper training
- ⑥ Practice-problem design
- ⑦ Formula and concept notes
- ⑧ 75-minute examination practice
- ⑨ 60-minute pressure-training mocks
- ⑩ Error analysis
- ⑪ Mathematical maturity
- ⑫ Confidence and consistency
- ⑬ STEP and harder mathematical preparation
- ⑭ University entrance examinations
- ⑮ Mathematical interviews
- ⑯ Preparing for mathematics-heavy university programmes

The Current TMUA Structure

TMUA consists of two papers.

Paper 1: Applications of Mathematical Knowledge

and

Paper 2: Mathematical Reasoning

Each paper contains

20 multiple-choice questions

and has an official duration of

75 minutes.

The Time Reality

For 20 questions in 75 minutes,

$$\frac{75}{20} = 3.75.$$

So the average available time is

3 minutes 45 seconds per question.

But questions are not equally difficult.

Therefore good time management does **not** mean spending exactly 3 minutes 45 seconds on every problem.

TMUA Is Multiple Choice — But Not Easy

Multiple choice does not mean simple mathematics.

The challenge may lie in:

- recognising the relevant idea;
- translating the problem correctly;
- eliminating impossible cases;
- detecting a hidden structure;
- avoiding unnecessary algebra;
- distinguishing very similar alternatives;
- reasoning quickly under time pressure.

Key Idea

The question is often not “Can you calculate this?”
It is “Can you see what matters?”

TMUA Paper 1

Paper 1 focuses on

Applications of Mathematical Knowledge.

The mathematical content is largely familiar school mathematics.
But questions may place familiar mathematics into unfamiliar situations.
Therefore the central skill is

transfer of mathematical knowledge.

What Does “Application” Mean?

Suppose you know the quadratic formula.
That is mathematical knowledge.
But a Paper 1 question may instead require you to recognise that:

$$b^2 - 4ac < 0$$

immediately implies

no real roots.

No quadratic formula is necessary.
The stronger student chooses the smallest sufficient tool.

Paper 1 Is Not a Formula Test

A weaker approach is:

Question $\longrightarrow$ Which formula do I remember?

A stronger approach is:

Question $\longrightarrow$ What mathematical structure is present?

$\longrightarrow$ What information is actually required?

$\longrightarrow$ What is the shortest valid route?

Paper 1 Skills

Develop the ability to:

- manipulate algebra accurately;
- recognise useful substitutions;
- interpret graphs;
- exploit symmetry;
- identify restrictions and domains;
- use inequalities intelligently;
- connect algebra and geometry;
- use calculus structurally;
- estimate when exact calculation is unnecessary;
- eliminate answer choices logically.

A Paper 1 Habit

Before calculating, ask:

What do I already know?

Then:

What exactly am I being asked?

Then:

What connects these two?

Only after that should substantial calculation begin.

TMUA Paper 2

Paper 2 focuses on

Mathematical Reasoning.

It also involves elementary ideas from logic and proof.

The student must become comfortable with statements such as:

$$P \Rightarrow Q,$$

necessary conditions,
sufficient conditions,
negation,
counterexamples,
and logical consequences.

Knowing a Statement vs Understanding Its Logic

Suppose

$$P \Rightarrow Q.$$

This does **not** automatically imply

$$Q \Rightarrow P.$$

However, it does imply the contrapositive

$$\neg Q \Rightarrow \neg P.$$

Paper 2 rewards students who understand the structure of such statements rather than merely recognise terminology.

Counterexamples Are Powerful

Suppose someone claims:

“For every real x , $P(x)$ is true.”

To disprove the claim, we do not need to analyse every real number.
We need only find

one valid counterexample.

This is one of the most efficient tools in mathematical reasoning.

Paper 2 Questions to Ask

When reading an argument, ask:

- ❶ What is being assumed?
- ❷ What is being concluded?
- ❸ Does the conclusion necessarily follow?
- ❹ Is the implication being reversed?
- ❺ Is a necessary condition being treated as sufficient?
- ❻ Can I construct a counterexample?
- ❼ Are all cases covered?
- ❽ Has an unstated assumption been introduced?

Paper 1 and Paper 2 Are Connected

Do not prepare for the two papers as completely separate worlds.

Paper 1 requires reasoning.

Paper 2 requires mathematics.

The difference is emphasis.

A mature TMUA candidate develops

mathematical knowledge + mathematical logic.

The Problem-Solving Cycle

When facing an unfamiliar problem:

Understand



Represent



Explore



The Problem-Solving Cycle (continued)

Choose



Execute



Check .

Step 1: Understand

Before solving, identify:

- What is given?
- What is unknown?
- What restrictions exist?
- Is the question asking for a value?
- A number of solutions?
- A statement that must be true?
- A statement that could be true?
- A graphical interpretation?

Misreading the logical task can destroy an otherwise correct solution.

Step 2: Represent

Translate the problem into something useful.

Possible representations include:

equation

diagram

graph

table

sequence

inequality

Step 2: Represent (continued)

logical statement .

Representation often determines difficulty.

Step 3: Explore Before Committing

Try small questions:

- What happens in a simple case?
- What happens at the boundary?
- Is there symmetry?
- Is anything invariant?
- Can I factor?
- Can I substitute?
- Can I draw it?
- Can I eliminate choices?
- Can I estimate?

Exploration is not wasted time if it reveals structure.

Step 4: Choose the Route

Several methods may work.

But under examination conditions, ask:

Which method gives enough information most efficiently?

Do not solve a stronger problem than the one you were asked.

For example, if the question asks only for the number of roots, you may not need the roots themselves.

Step 5: Execute Cleanly

Once a route is selected:

- keep algebra organised;
- avoid unnecessary expansion;
- track signs carefully;
- respect domains;
- preserve equivalent statements;
- simplify strategically.

Good problem solving still requires reliable technical execution.

Step 6: Check

Before committing to an answer:

- Does the sign make sense?
- Is the value in the allowed domain?
- Did I introduce an extraneous solution?
- Did I answer the actual question?
- Does the result agree with the graph?
- Is the magnitude reasonable?

A ten-second check can save a question.

Do Not Stop at “How”

Suppose you learn a method.

A basic learner asks:

How do I do it?

A stronger learner asks:

Why does it work?

A developing mathematician also asks:

What if something changes?

That final question is extremely important.

Example of the What-If Habit

Suppose

$$ax^2 + bx + c = 0.$$

Do not merely know

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Ask:

- Why must $a \neq 0$?
- What if $b^2 - 4ac = 0$?
- What if $b^2 - 4ac < 0$?
- What changes geometrically?
- What if the quadratic is always positive?
- What if a parameter replaces one coefficient?

That is deeper preparation.

Five Questions for Every Important Result

For every major theorem, formula or method, ask:

1. *What does it say?*

2. *Why does it work?*

3. *When can I use it?*

4. *When can I not use it?*

5. *What changes if the assumptions change?*

These questions build transferable mathematical understanding.

Past Papers Are Essential

Past papers reveal:

- the style of questions;
- the expected depth;
- common structures;
- recurring reasoning patterns;
- realistic time pressure;
- your personal weaknesses.

But simply completing many papers is not enough.

Key Idea

Past papers should be studied, not merely consumed.

The Wrong Way to Use Past Papers

A weak cycle is:



This produces experience,
but may produce surprisingly little improvement.

The Better Past-Paper Cycle

Use:

Attempt



Review



Diagnose



The Better Past-Paper Cycle (continued)

Relearn



Re-solve



Generalise

This turns each question into a training resource.

After Every Difficult Question

Ask:

- ❶ Why did I get stuck?
- ❷ What idea unlocked the problem?
- ❸ Could I recognise that idea earlier next time?
- ❹ Was there a shorter solution?
- ❺ Which clue did I miss?
- ❻ Was the problem conceptual or computational?
- ❼ Can I create a similar problem myself?

The review may be more valuable than the original attempt.

Not All Wrong Answers Are the Same

Classify mistakes.

For example:

K: Knowledge Gap

R: Reasoning Error

A: Algebra / Arithmetic Error

M: Misread Question

T: Time-Management Error

S: Strategy Error

Different errors require different remedies.

Knowledge Gap

Example:

You did not know an identity or theorem needed for the problem.

Solution:

Return to concept revision.

Do not try to repair a genuine knowledge gap merely by doing more timed papers.

Reasoning Error

Perhaps you knew all the mathematics,
but failed to connect the ideas.

Solution:

- study the solution carefully;
- identify the turning point;
- explain the reasoning in your own words;
- find related problems;
- re-solve the question later.

This is how problem-solving ability develops.

Execution Error

You may have had the correct strategy but made an error such as

$$-(x - 3)$$

becoming

$$-x - 3$$

instead of

$$-x + 3.$$

This is not a conceptual weakness.

It indicates an accuracy or execution problem.

The remedy is slower, cleaner mathematical writing during training.

Strategy Error

Perhaps you used a valid method,
but it required two pages of algebra.

The official-style solution may reveal a structural observation that solves the problem in a few lines.

Ask:

Could I have seen the structure earlier?

Efficiency itself is a trainable mathematical skill.

Practice Beyond Past Papers

Past papers are finite.

They should not be exhausted too early.

Use additional problems to build skills in:

- algebraic manipulation;
- functions and graphs;
- sequences;
- geometry;
- trigonometry;
- logarithms;
- calculus;
- logic;
- proof;
- mixed problem solving.

Three Levels of Practice

A useful progression is:

Level A: Foundation

Concept and technique.

Level B: TMUA Standard

Application and reasoning.

Level C: Advanced Application

Unfamiliar combinations, deeper reasoning and harder adaptations.

Do Not Rush Through Easy Questions

Easy and medium problems are not useless.

They develop:

- fluency;
- recognition;
- accuracy;
- speed;
- confidence.

But once a skill becomes automatic, practice should become progressively more demanding.

Quality of Practice Matters

Compare:

1000

questions solved mechanically
with

300

questions studied deeply.

The second programme may produce greater improvement if the student:

- analyses errors;
- compares methods;
- revisits difficult questions;
- generalises ideas;
- learns from every problem.

Build Detailed Mathematical Notes

Create a compact but meaningful revision system.

For each topic include:

- definitions;
- important results;
- formulae;
- conditions and restrictions;
- graphical interpretations;
- common transformations;
- common mistakes;
- useful problem-solving observations.

A Formula Without Conditions Is Dangerous

For example,

$$S_{\infty} = \frac{a}{1-r}$$

is incomplete unless you remember

$$\boxed{|r| < 1}.$$

Similarly,

$$\log_a x$$

requires appropriate restrictions such as

$$a > 0, \quad a \neq 1, \quad x > 0.$$

Good notes record the conditions, not merely the formula.

Formula Notes Should Include Meaning

Do not write only

$$b^2 - 4ac.$$

Write:

$$\Delta = b^2 - 4ac$$

together with

$\Delta > 0 \Rightarrow$ 2 distinct real roots,

$\Delta = 0 \Rightarrow$ repeated root,

$\Delta < 0 \Rightarrow$ no real roots.

Then connect these to the graph.

Build a Problem-Solving Notebook Too

Your formula notes answer:

What do I need to know?

Your problem-solving notebook answers:

What have difficult problems taught me?

Record observations such as:

- useful substitutions;
- symmetry tricks;
- hidden factorisations;
- graphical interpretations;
- clever inequalities;
- logical traps;
- alternative methods.

Official Timing vs Training Timing

The official TMUA timing is

75 minutes per paper.

Therefore students must practise full papers under genuine 75-minute conditions. However, during advanced preparation, I also recommend selected training mocks of

60 minutes.

These are deliberately harder training conditions.

Why Train Some Mocks in 60 Minutes?

A 60-minute training mock gives an average of

$$\frac{60}{20} = 3$$

minutes per question.

This creates additional pressure.

The purpose is to train:

- rapid recognition;
- decision making;
- question selection;
- emotional control;
- recovery after difficult questions;
- efficient algebra.

Important: 60 Minutes Is Training

Do Not Confuse Training With the Examination

The official paper duration is 75 minutes.

The 60-minute format is an optional preparation technique, not the official TMUA format.

The objective is analogous to athletic overtraining:

train under controlled pressure

so that normal examination conditions feel more manageable.

Do Not Start With 60-Minute Mocks

A student who is still learning the mathematics should not be forced immediately into extreme time pressure.

Use progression:

Untimed



75 minutes



70 minutes



Do Not Start With 60-Minute Mocks (continued)

65 minutes



selected 60-minute training mocks.

Then Return to 75 Minutes

Before the actual examination,
return to realistic simulation:

75 minutes.

The goal is not to condition yourself to rush every question.
The goal is to enter the examination with

time reserve + calmness + control.

Do Not Fight One Question for Too Long

One difficult problem can consume

8, 10, 12

minutes.

That can damage the rest of the paper.

A disciplined candidate recognises:

I may be able to solve this, but not efficiently right now.

Move on and return later if time permits.

A Three-Pass Strategy

One possible approach:

Pass 1

Solve questions whose route is immediately visible.

Pass 2

Return to medium questions requiring more work.

Pass 3

Use remaining time for the hardest unresolved questions and checking.

But Do Not Follow Timing Rules Blindly

Different students have different strengths.

One candidate may be exceptionally fast in algebra.

Another may be stronger in geometry or logical reasoning.

Therefore your timing strategy should be informed by data from your own mocks.

Strategy

Build a strategy from evidence about your performance, not from arbitrary rules.

What Is Mathematical Maturity?

Mathematical maturity is difficult to reduce to one definition.

It includes the ability to:

- reason from definitions;
- recognise structure;
- connect different topics;
- distinguish examples from proofs;
- detect hidden assumptions;
- generalise;
- construct counterexamples;
- choose efficient representations;
- tolerate unfamiliarity;
- persist productively.

Mathematical Maturity Changes the Question

A beginner asks:

Which formula should I use?

A stronger student asks:

Which topic is this?

A mathematically mature student increasingly asks:

What structure does this problem have?

That shift is extremely important.

From Procedure to Structure

Suppose

$$x^2 + y^2 = 10$$

and

$$x + y = 4.$$

A purely procedural approach may immediately substitute.

A structural thinker may notice

$$(x + y)^2 = x^2 + y^2 + 2xy.$$

Therefore

$$16 = 10 + 2xy,$$

so

From Procedure to Structure (continued)

$$xy = 3.$$

The problem becomes simpler because structure was recognised.

Mathematical Maturity Means Asking Why

When learning a result, do not stop at

I can use it.

Aim for

I understand why it works.

Then progress to

I know when it fails.

and finally

I can adapt the idea to something unfamiliar.

Learn to Be Comfortable With Being Stuck

Hard mathematics often begins with

I do not immediately know what to do.

This is normal.

Productive persistence means:

- trying representations;
- examining special cases;
- simplifying;
- looking for patterns;
- testing conjectures;
- abandoning an unproductive route;
- returning with a different idea.

Confidence Is Built, Not Declared

Useful mathematical confidence comes from evidence.

Conceptual Understanding

+

Repeated Practice

+

Successful Recovery From Mistakes

$\Rightarrow$

Reliable Confidence

Confidence Does Not Mean Knowing Everything

A confident problem solver can still encounter a question and think:

I do not see the solution yet.

The important word is

yet.

Confidence means trusting the process of mathematical exploration rather than expecting immediate recognition every time.

Consistency Beats Occasional Intensity

Compare:

7 hours once every two weeks

with

45–90 minutes of focused work regularly.

For most students, sustained practice produces better retention, recognition and fluency.

Key Idea

Mathematical ability compounds through repeated contact with good problems.

Phase 1: Build the Foundation

First:

- study the specification;
- identify weak topics;
- revise mathematical concepts;
- create formula/concept notes;
- solve topic-based exercises.

At this stage:

accuracy before speed .

Phase 2: Develop Problem Solving

Next:

- solve unfamiliar problems;
- mix topics;
- compare alternative methods;
- practise graph interpretation;
- develop logical reasoning;
- maintain an error notebook.

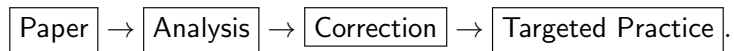
Now the objective becomes:

flexibility rather than routine repetition.

Phase 3: Past-Paper Training

Now begin systematic past-paper work.

Use:



Do not measure progress only by the raw number of papers completed.

Measure whether repeated weaknesses are disappearing.

Phase 4: Timed Performance

Introduce:

- full 75-minute papers;
- realistic computer-based practice where available;
- strict no-calculator conditions;
- question-selection strategy;
- selected shortened pressure mocks.

Now the objective is

reliable mathematics under pressure.

Phase 5: Final Consolidation

Near the examination:

- revisit recurring errors;
- revise formula/concept notes;
- re-solve selected difficult questions;
- complete realistic timed papers;
- maintain normal sleep and study routines;
- avoid uncontrolled last-minute topic expansion.

Preparation should become increasingly stable.

Do Not Prepare Only for One Examination

The deeper objective is not merely

TMUA preparation.

It is to develop the mathematics needed for

demanding university-level quantitative study.

That requires more than memorised techniques.

TMUA and STEP Are Different

TMUA emphasises rapid mathematical reasoning in a multiple-choice environment.
STEP requires substantially longer written solutions to harder problems.
Therefore preparation differs.
TMUA asks, among other things:

Can you recognise and reason efficiently?

STEP places much more emphasis on:

Can you develop a substantial mathematical solution?

Preparing for STEP

For STEP-style mathematics, develop:

- deeper algebraic fluency;
- proof;
- multi-stage problem solving;
- persistence;
- careful written reasoning;
- creative substitutions;
- connections across topics;
- the ability to make partial progress.

A hard question may require several ideas before the final structure becomes visible.

STEP Rewards Progress

For long-form mathematics, an important skill is making useful progress even before the complete solution is visible.

You may:

- establish an intermediate identity;
- prove a special case;
- transform the expression;
- derive a useful bound;
- introduce an appropriate substitution.

The mathematical journey itself matters.

Different Universities Use Different Processes

Highly selective universities may use different combinations of:

- school qualifications;
- admissions tests;
- written applications;
- interviews;
- contextual information;
- other programme-specific requirements.

Requirements vary by university, programme and admissions cycle.
Always verify the current requirements for the exact course.

Mathematics-Heavy Programmes

Strong mathematical preparation can be especially valuable for programmes such as:

Mathematics

Engineering

Computer Science

Statistics

Economics

Physics

Mathematics-Heavy Programmes (continued)

Finance and quantitative disciplines.

The exact admissions process differs across institutions.

Prestigious Mathematics Environments

Students aiming for highly selective mathematics-intensive programmes may encounter very strong competition.

The correct response is not merely to accumulate more formulae.

Develop:

depth

precision

reasoning

independence

mathematical curiosity.

UK and International Applications

For applicants considering leading UK, US or other international universities:

Key Idea

There is no single universal mathematics admissions test.

Different universities and programmes use different requirements.

Therefore:

check the current official course requirements

before constructing an examination-preparation plan.

An Interview Is Not Just an Oral Exam

In a mathematical interview or interactive problem-solving setting, the interviewer may be interested in how you respond to an unfamiliar problem.

Useful habits include:

- thinking aloud clearly;
- explaining assumptions;
- testing examples;
- responding constructively to hints;
- correcting mistakes;
- justifying claims;
- exploring alternative approaches.

Do Not Pretend to Know

Suppose you are stuck.

Instead of guessing randomly, explain what you know.

For example:

“I do not yet see the general argument,

but I would first examine the cases $n = 1, 2, 3$

to see whether a pattern emerges.”

This demonstrates a mathematical process.

Responding to a Hint

If an interviewer suggests:

“What happens if you differentiate it?”

do not treat the hint merely as an instruction.

Ask:

Why might differentiation reveal useful structure?

Use the hint to advance your understanding of the problem.

Explain Why

Avoid answers of the form:

“because that is the formula.”

Try to explain:

- why the formula applies;
- what assumptions are being used;
- what the result means;
- whether another method exists;
- what would change in a modified problem.

This is useful both for interviews and for advanced mathematics.

The Six Levels of Understanding

For an important mathematical idea, aim to reach:

1. Recall

2. Execute

3. Explain

4. Justify

5. Adapt

6. Generalise.

This is far more robust than memorisation alone.

Level 1: Recall

Can you state the result?

For example,

$$\sin^2 x + \cos^2 x = 1.$$

This is necessary.

But it is only the beginning.

Level 2: Execute

Can you use the identity correctly?
For example,

$$\sin^2 x = 1 - \cos^2 x.$$

Can you recognise when that substitution simplifies a problem?
Now knowledge becomes usable.

Level 3: Explain

Can you explain what the identity means?
Can you connect it to

$$x^2 + y^2 = 1$$

on the unit circle?
Can you explain why sine and cosine are linked in this way?
Now the result is connected to geometry.

Level 4: Justify

Can you derive or prove the result from a mathematical model?
Can you identify the assumptions involved?
This moves beyond acceptance toward mathematical reasoning.

Level 5: Adapt

Suppose the problem changes.
Can you use

$$\sin^2 x + \cos^2 x = 1$$

inside an unfamiliar equation, inequality or optimisation problem?
This is the level admissions tests often demand.

Level 6: Generalise

Can you ask:

What larger mathematical idea is present?

Can the relationship be represented differently?

Can it connect to vectors, complex numbers, geometry or calculus?

This is where mathematical maturity begins to become visible.

Do Not Chase Tricks

Shortcuts can be useful.

But a preparation programme built entirely around tricks is fragile.

A stronger hierarchy is:

Understanding



Technique



Efficiency



Do Not Chase Tricks (continued)

Speed.

Speed should emerge from mastery.

Build a Mathematical Toolkit

Your toolkit should contain:

- algebraic manipulation;
- graphical reasoning;
- geometric insight;
- logical reasoning;
- estimation;
- substitution;
- symmetry;
- inequalities;
- calculus;
- proof ideas;
- counterexamples;
- strategic use of special cases.

But the real skill is knowing which tool to choose.

The Strong Candidate

A strong candidate does not necessarily calculate fastest.

They often:

- recognise structure earlier;
- avoid unnecessary work;
- recover quickly from mistakes;
- know when to leave a question;
- remain mathematically organised;
- reason reliably under pressure.

That combination produces examination efficiency.

The Long-Term Goal

The deepest objective is not:

become good at TMUA questions.

It is:

become a stronger mathematical problem solver.

Then TMUA becomes one environment in which that ability is tested.

STEP, university mathematics and mathematical interviews become further environments in which the same deeper ability matters.

TMUA Preparation Checklist

Before the examination, ask:

- 1 Do I know the specification thoroughly?
- 2 Are my mathematical foundations secure?
- 3 Can I explain important results?
- 4 Can I recognise common structures quickly?
- 5 Have I studied Paper 1 problems systematically?
- 6 Have I studied Paper 2 reasoning systematically?
- 7 Have I completed official past/practice material?
- 8 Have I analysed my errors?
- 9 Have I practised under 75-minute conditions?
- 10 Have I developed a reliable timing strategy?
- 11 Can I remain composed when a problem is unfamiliar?

Advanced Preparation Checklist

For students aiming beyond TMUA:

- 1 Can I solve longer unfamiliar problems?
- 2 Can I construct mathematical arguments?
- 3 Can I explain why methods work?
- 4 Can I identify necessary assumptions?
- 5 Can I generalise from examples?
- 6 Can I find counterexamples?
- 7 Can I respond productively to hints?
- 8 Can I communicate mathematics clearly?
- 9 Can I persist when the solution is not immediate?
- 10 Can I learn from unsuccessful approaches?

Confidence + Consistency

A single excellent study session is useful.

But mathematical development comes from

Confidence + Consistency

built on genuine understanding.

The Complete Preparation Model

Learn



Understand

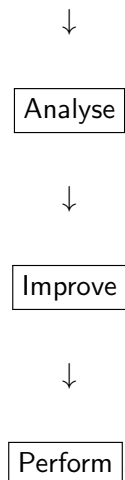


Practise



Struggle

The Complete Preparation Model (continued)



From TMUA to University Mathematics

TMUA preparation can be more than examination preparation.

It can begin the transition from

School Mathematics



Advanced Problem Solving



Mathematical Maturity



University Mathematics

Final Message

**Do not prepare only to recognise questions.
Prepare to think when the question is unfamiliar.**

Understand deeply

Practise intelligently

Analyse mistakes

Final Message (continued)

Build mathematical maturity

Confidence and Consistency

are built through preparation.

Know → Understand → Reason → Solve → Perform

Mathematics Elevate Academy

Rishabh Kumar