JEE Advanced 2026 Paper 1

Mathematics Detailed Solutions

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Premium Conceptual Solutions for JEE Advanced Aspirants

2026 Edition

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Mathematics Detailed Solutions for JEE Advanced 2026 Paper 1

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Math By Rishabh Series

Mathematics Elevate Series

Mathematics Elevate Academy

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About This Solution Booklet

This booklet is designed for serious JEE Advanced aspirants seeking conceptual clarity, mathematical maturity, and elegant problem-solving techniques.

The solutions are intentionally written beyond conventional coaching style. The emphasis is placed on:

- rigorous mathematical reasoning,
- elegant simplifications,
- strategic thinking,
- advanced exam insights,
- and Olympiad-style observations.

Question 1 : Analysis of a Logarithmic-Root Function

Problem Statement

Consider the function

$$f : (0, \infty) \longrightarrow \mathbb{R}$$

defined by

$$f(x) = \sqrt{x}\ln x - x + 1.$$

Determine the correct statement.

Difficulty:
Theme: Monotonicity and Critical Points
Target Time: 4 minutes

Core Mathematical Idea

The problem is fundamentally about sign analysis of the first derivative.
A stationary point does not necessarily imply a local extremum. The key objective is to determine whether $f'(x)$ changes sign.

Solution

Differentiate:

$$f(x) = \sqrt{x}\ln x - x + 1.$$

Using the product rule,

$$\frac{d}{dx}\left(\sqrt{x}\ln x\right) = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}.$$

Hence,

$$f'(x) = \frac{\ln x + 2 - 2\sqrt{x}}{2\sqrt{x}}.$$

Define

$$g(x) = \ln x + 2 - 2\sqrt{x}.$$

Then

$$f'(x) = \frac{g(x)}{2\sqrt{x}}.$$

Therefore, the sign of $f'(x)$ is determined entirely by $g(x)$.

Now,

$$g'(x) = \frac{1}{x} - \frac{1}{\sqrt{x}} = \frac{1-\sqrt{x}}{x}.$$

Thus:

$$g'(x) > 0 \quad \text{for } 0 < x < 1,$$

and

$$g'(x) < 0 \quad \text{for } x > 1.$$

Consequently, $g(x)$ attains its maximum value at $x = 1$.
Evaluating,

$$g(1) = \ln 1 + 2 - 2 = 0.$$

Hence,

$$g(x) \leq 0 \quad \forall x > 0.$$

Therefore,

$$f'(x) \leq 0 \quad \forall x > 0.$$

Since $f'(x)$ never changes sign, the function is monotone decreasing on $(0, \infty)$.
Accordingly, the function possesses neither a local maximum nor a local minimum.

Rishabh's Insight

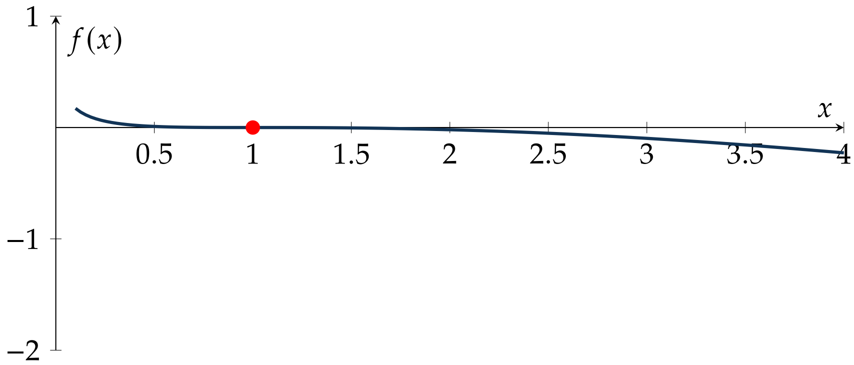
The point $x = 1$ is a stationary point but not an extremum.
This distinction is one of the most frequently misunderstood ideas in differential calculus.

Exam Trap

Many students incorrectly conclude that $f'(1) = 0$ automatically implies a maximum or minimum.
A sign change analysis is essential.

Final Answer

$$\boxed{\text{Option (D)}}$$



Question 2 : Circumradius from Tangent Geometry

Problem Statement

Point P lies on the parabola

$$y = x^2$$

such that the tangent slope at P equals 4.

Point Q lies in the first quadrant on the circle

$$x^2 + y^2 = 2$$

such that the tangent slope at Q equals -1.

Point R lies in the first quadrant on the ellipse

$$x^2 + 4y^2 = 8$$

such that the tangent slope at R equals $-\frac{1}{2}$.

Find the radius of the circumcircle through P, Q, R.

Difficulty:
Theme: Coordinate Geometry + Tangents
Target Time: 3 minutes

Core Mathematical Idea

The slopes determine the coordinates directly via implicit differentiation.
After obtaining the coordinates, observe the hidden right triangle structure.

Step 1 : Determination of Point P

For

$$y = x^2,$$

we have

$$\frac{dy}{dx} = 2x.$$

Given slope = 4,

$$2x = 4 \quad \Rightarrow \quad x = 2.$$

Hence,

$$y = 2^2 = 4.$$

Therefore,

$$P = (2, 4).$$

Step 2 : Determination of Point Q

For

$$x^2 + y^2 = 2,$$

implicit differentiation yields

$$2x + 2y\frac{dy}{dx} = 0.$$

Thus,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Given slope $= -1$,

$$-\frac{x}{y} = -1 \quad \Rightarrow \quad x = y.$$

Substituting into the circle equation,

$$2x^2 = 2.$$

Hence,

$$x = 1, \quad y = 1.$$

Therefore,

$$Q = (1, 1).$$

Step 3 : Determination of Point R

For

$$x^2 + 4y^2 = 8,$$

implicit differentiation gives

$$2x + 8y\frac{dy}{dx} = 0.$$

Hence,

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

Given slope $= -\frac{1}{2}$,

$$-\frac{x}{4y} = -\frac{1}{2}.$$

Therefore,

$$x = 2y.$$

Substituting into the ellipse equation,

$$(2y)^2 + 4y^2 = 8$$

$$8y^2 = 8.$$

Thus,

$$y = 1, \quad x = 2.$$

Therefore,

$$R = (2, 1).$$

Pro Shortcut

Observe immediately that:

$$PR \text{ is vertical}$$

and

$$QR \text{ is horizontal.}$$

Hence,

$$\angle PRQ = 90^\circ.$$

For a right triangle, the circumradius equals half the hypotenuse.

The hypotenuse is

$$PQ = \sqrt{(2-1)^2 + (4-1)^2} = \sqrt{10}.$$

Therefore, the circumradius is

$$\frac{\sqrt{10}}{2} = \sqrt{\frac{5}{2}}.$$

Rishabh's Insight

Many coordinate geometry problems become dramatically simpler after identifying hidden perpendicular structures.

Final Answer

$$\boxed{\text{Option (C)}}$$

Question 3 : Elementary Row Transformations and Invertibility

Problem Statement

Which one of the following matrices can be obtained by performing elementary row transformations on the 3×3 identity matrix?

Difficulty:
Theme: Matrices and Elementary Operations
Target Time: 2 minutes

Core Mathematical Idea

Elementary row operations preserve invertibility.
Hence, any matrix obtainable from the identity matrix through elementary row operations must necessarily be non-singular.

Solution

The identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible with determinant

$$\det(I_3) = 1 \neq 0.$$

Every elementary row transformation corresponds to multiplication by an elementary matrix, which is itself invertible.

Consequently, any matrix obtained from I_3 via elementary row transformations must also be invertible.

Therefore, the problem reduces to identifying the matrix with non-zero determinant.

Checking Option (A)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

All three rows are identical.
Hence,

$$\det(A) = 0.$$

Therefore, A is singular.

Checking Option (B)

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

Evaluating determinant:

$$\det(B) = 1(3 - 8) - 1(2 - 4) + 1(4 - 3).$$

Thus,

$$\det(B) = -5 + 2 + 1 = -2.$$

Since

$$\det(B) \neq 0,$$

the matrix is invertible.

Checking Option (C)

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

Evaluating determinant:

$$\det(C) = 1(24 - 20) - 1(16 - 8) + 1(10 - 6).$$

Hence,

$$\det(C) = 4 - 8 + 4 = 0.$$

Thus, C is singular.

Checking Option (D)

$$D = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

Evaluating determinant:

$$\det(D) = 1(3 - 4) - 1(-3 - 0) + 1(-2 - 0).$$

Thus,

$$\det(D) = -1 + 3 - 2 = 0.$$

Hence, D is singular.

Rishabh's Insight

This question is conceptually testing a deep linear algebra principle:

Elementary row operations preserve rank and invertibility.

A quick determinant check is the fastest competitive approach.

Pro Shortcut

In JEE Advanced, whenever a matrix is obtained from the identity matrix using elementary operations, immediately think:

$$\boxed{\det \neq 0}$$

This eliminates most options within seconds.

Final Answer

$$\boxed{\text{Option (B)}}$$

Question 4 : Principal Values in Inverse Trigonometric Functions

Problem Statement

Evaluate

$$\cot^{-1}(\cot(-11)) + 10\sin\left(2\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) + 10\sin\left(2\tan^{-1}(2)\right).$$

Difficulty:
Theme: Inverse Trigonometric Functions
Target Time: 4 minutes

Core Mathematical Idea

The most important concept is the principal value range of inverse trigonometric functions. For $\cot^{-1} x$, the principal value lies in

$$(0, \pi).$$

Step 1 : Evaluation of $\cot^{-1}(\cot(-11))$

We seek the unique angle

$$\theta \in (0, \pi)$$

such that

$$\cot\theta = \cot(-11).$$

Using periodicity,

$$\cot(\theta) = \cot(\theta + n\pi).$$

Hence,

$$\theta = -11 + n\pi.$$

We now determine n such that

$$0 < -11 + n\pi < \pi.$$

Taking $n = 4$,

$$-11 + 4\pi = 4\pi - 11.$$

Since

$$0 < 4\pi - 11 < \pi,$$

we obtain

$$\cot^{-1}(\cot(-11)) = 4\pi - 11.$$

Step 2 : Evaluation of the Second Term

Let

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

Since

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

we have

$$\theta = \frac{\pi}{4}.$$

Therefore,

$$10\sin(2\theta) = 10\sin\left(\frac{\pi}{2}\right) = 10.$$

Step 3 : Evaluation of the Third Term

Let
$$\phi = \tan^{-1}(2).$$

Using the identity
$$\sin(2\phi) = \frac{2\tan\phi}{1+\tan^2\phi},$$
we obtain
$$\sin(2\phi) = \frac{2(2)}{1+2^2} = \frac{4}{5}.$$

Hence,
$$10\sin(2\phi) = 10 \cdot \frac{4}{5} = 8.$$

Final Computation

Combining all terms,
$$(4\pi - 11) + 10 + 8 = 4\pi + 7.$$

Rishabh's Insight

Inverse trigonometric questions in JEE Advanced are often disguised tests of principal value ranges rather than trigonometric identities themselves.

Exam Trap

A common mistake is writing
$$\cot^{-1}(\cot x) = x.$$
This is false unless
$$x \in (0, \pi).$$
Always reduce the angle into the principal branch.

Pro Shortcut

Whenever expressions of the form
$$\sin(2\tan^{-1} x)$$
appear, immediately use:
$$\sin(2\tan^{-1} x) = \frac{2x}{1+x^2}.$$

Final Answer

$$\boxed{\text{Option (C)}}$$

Question 5 : Probability and Independence of Events

Problem Statement

Box I contains 6 Red balls and 9 Green balls.
Box II contains 8 Red balls and 12 Green balls.
All balls are mixed together.

Define the events:

$$E_1: \text{ ball originated from Box I,}$$
$$E_2: \text{ ball originated from Box II,}$$
$$F_1: \text{ selected ball is Red,}$$
$$F_2: \text{ selected ball is Green.}$$

Determine the correct statements.

Difficulty:
Theme: Conditional Probability and Independence
Target Time: 3 minutes

Core Mathematical Idea

If the colour ratios in both boxes are identical, knowledge of the originating box provides no additional information about the colour of the selected ball.
This naturally suggests independence.

Step 1 : Conditional Probabilities

For Box I,

$$P(F_1|E_1) = \frac{6}{15} = \frac{2}{5}.$$

For Box II,

$$P(F_1|E_2) = \frac{8}{20} = \frac{2}{5}.$$

Thus,

$$P(F_1|E_1) = P(F_1|E_2).$$

Step 2 : Probability of Selecting a Red Ball

Total number of balls:

$$15 + 20 = 35.$$

Hence,

$$P(E_1) = \frac{15}{35} = \frac{3}{7}, \qquad P(E_2) = \frac{20}{35} = \frac{4}{7}.$$

Using the law of total probability,

$$P(F_1) = P(F_1|E_1)P(E_1) + P(F_1|E_2)P(E_2).$$

Therefore,

$$P(F_1) = \frac{2}{5} \cdot \frac{3}{7} + \frac{2}{5} \cdot \frac{4}{7} = \frac{2}{5}.$$

Consequently,

$$P(F_1|E_1) = P(F_1).$$

Hence,

$$E_1 \text{ and } F_1 \text{ are independent.}$$

Therefore, option (A) is correct.

Step 3 : Analysis of Option (B)

For Green balls,

$$P(F_2|E_2) = \frac{12}{20} = \frac{3}{5}.$$

Similarly,

$$P(F_2) = \frac{3}{5}.$$

Thus,

$$P(F_2|E_2) = P(F_2),$$

showing that E_2 and F_2 are also independent.
Therefore, option (B) is false.

Step 4 : Analysis of Option (D)

We compare:

$$P(F_1|E_1) = \frac{2}{5}$$

and

$$P(F_2|E_2) = \frac{3}{5}.$$

Since

$$\frac{2}{5} < \frac{3}{5},$$

option (D) is false.

Rishabh's Insight

Whenever two groups possess identical ratios, conditional probabilities often become equal, leading naturally to independence.

Pro Shortcut

Observe immediately:

$$6 : 9 = 8 : 12 = 2 : 3.$$

Hence, the proportion of Red balls is identical in both boxes.
This instantly suggests independence.

Final Answer

$$\boxed{\text{Options (A) and (C)}}$$

Question 6 : Plane Through a Line in Three-Dimensional Geometry

Problem Statement

A plane P contains the line

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{1}$$

and is perpendicular to the plane

$$x+2y+3z=4.$$

Another plane P_1 passes through the point

$$(4,2,2)$$

and is parallel to P.

Determine the correct statements.

Difficulty:
Theme: Three-Dimensional Geometry
Target Time: 5 minutes

Core Mathematical Idea

The normal vector of the required plane must be perpendicular to:

(i) the direction vector of the line

and

(ii) the normal vector of the given plane.

Hence, the cross product naturally determines the required normal vector.

Step 1 : Direction and Normal Vectors

The given line has direction vector

$$\vec{d}=(2,3,1).$$

The plane

$$x+2y+3z=4$$

has normal vector

$$\vec{n}_1=(1,2,3).$$

The normal vector of plane P must be perpendicular to both vectors.
Hence,

$$\vec{n}=\vec{d}\times\vec{n}_1.$$

Computing,

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 1 & 2 & 3\end{vmatrix}.$$

Therefore,

$$\vec{n}=(7,-5,1).$$

Step 2 : Equation of Plane P

The line passes through

$$(1, 3, -2).$$

Hence, the required plane is

$$7(x-1) - 5(y-3) + (z+2) = 0.$$

Simplifying,

$$7x - 5y + z = -10.$$

Thus, option (A) is correct.

Step 3 : Equation of Plane P_1

Since $P_1 \parallel P$, both planes possess the same normal vector.

Thus,

$$7x - 5y + z = \lambda.$$

Substituting the point $(4, 2, 2)$,

$$7(4) - 5(2) + 2 = 20.$$

Hence,

$$P_1 : \quad 7x - 5y + z = 20.$$

Step 4 : Distance Calculations

Distance of plane P from the origin:

$$\frac{|-10|}{\sqrt{7^2 + (-5)^2 + 1^2}} = \frac{10}{\sqrt{75}} = \frac{2}{\sqrt{3}}.$$

Hence, option (D) is correct.

Distance between P and P_1:

$$\frac{|20 - (-10)|}{\sqrt{75}} = \frac{30}{5\sqrt{3}} = 2\sqrt{3}.$$

Rishabh's Insight

In three-dimensional geometry, cross products frequently appear whenever a vector must simultaneously satisfy two perpendicularity conditions.

Exam Trap

A common mistake is taking the cross product in the wrong order or making sign errors during determinant expansion.

Final Answer

$$\boxed{\text{Options (A) and (D)}}$$

Question 7 : Continuity and Differentiability of $g(x) = xf(x)$

Problem Statement

Let

$$f : \mathbb{R} \to \mathbb{R}$$

be an arbitrary function and define

$$g(x) = xf(x).$$

Determine the correct statements concerning continuity and differentiability at $x = 0$.

Difficulty:
Theme: Continuity and Differentiability
Target Time: 5 minutes

Core Mathematical Idea

Multiplication by x does not automatically remove pathological behaviour of $f(x)$.
A careful limit-based analysis is essential.

Analysis of Option (A)

Consider the function

$$f(x) = \frac{1}{x^2}.$$

Then,

$$g(x) = x \cdot \frac{1}{x^2} = \frac{1}{x}.$$

Clearly,

$$\lim_{x \to 0} \frac{1}{x}$$

does not exist.

Hence, $g(x)$ is not continuous at $x = 0$.
Therefore, option (A) is false.

Analysis of Option (B)

Assume f is continuous at $x = 0$.
Then,

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}.$$

Since

$$g(h) = hf(h), \qquad g(0) = 0,$$

we get

$$g'(0) = \lim_{h \to 0} \frac{hf(h)}{h} = \lim_{h \to 0} f(h).$$

Because f is continuous at 0,

$$\lim_{h \to 0} f(h) = f(0).$$

Hence,

$$g'(0) = f(0),$$

which exists.

Therefore, g is differentiable at 0.
Thus, option (B) is true.

Analysis of Option (C)

Suppose g is differentiable at 0.

Then,

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} f(x).$$

Thus,

$$\lim_{x \to 0} f(x)$$

exists.

However, existence of the limit alone does not guarantee continuity of f at 0.

Consider the counterexample:

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 5, & x = 0. \end{cases}$$

Then,

$$g(x) = x f(x)$$

satisfies

$$g(x) = x \quad (x \neq 0), \qquad g(0) = 0.$$

Hence,

$$g'(0) = \lim_{h \to 0} \frac{h - 0}{h} = 1,$$

so g is differentiable at 0.

But,

$$\lim_{x \to 0} f(x) = 1 \neq f(0) = 5,$$

therefore f is not continuous at 0.

Hence, option (C) is false.

Analysis of Option (D)

From the previous derivation,

$$g'(0) = \lim_{x \to 0} f(x).$$

Therefore, if g is differentiable at 0, then

$$\lim_{x \to 0} f(x)$$

must exist.

Hence, option (D) is true.

Rishabh's Insight

The existence of a limit and continuity are different concepts.
Continuity additionally requires:

$$\lim_{x \to 0} f(x) = f(0).$$

Many advanced calculus questions exploit this subtle distinction.

Exam Trap

Students frequently confuse:

$$\lim_{x\to 0} f(x) \text{ exists}$$

with

$$f \text{ is continuous at } 0.$$

A function can possess a limit and still fail to be continuous if the function value differs from the limit.

Final Answer

$$\boxed{\text{Options (B) and (D)}}$$

Question 8 : Matrix Powers via Nilpotent Decomposition

Problem Statement

Let

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose

$$M^{26} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

and

$$\sum_{k=1}^{26} M^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Determine the correct statements.

Difficulty:
Theme: Matrix Algebra and Nilpotent Matrices
Target Time: 5 minutes

Core Mathematical Idea

Repeated eigenvalue problems often simplify dramatically through nilpotent decomposition. Try expressing the matrix in the form:

$$M = I + N,$$

where N is nilpotent.

Step 1 : Eigenvalue Structure

The characteristic polynomial is

$$\det(M - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix}.$$

Thus,

$$\lambda^2 - 2\lambda + 1 = 0.$$

Hence,

$$(\lambda - 1)^2 = 0.$$

Therefore, the only eigenvalue is

$$\lambda = 1.$$

Step 2 : Nilpotent Decomposition

Observe that

$$M - I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} =: N.$$

Hence,

$$M = I + N.$$

Now compute:

$$N^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, N is nilpotent.

Step 3 : Computation of M^k

Using the binomial theorem,

$$M^k = (I + N)^k.$$

Since

$$N^2 = 0,$$

all higher powers vanish.
Therefore,

$$M^k = I + kN.$$

Thus,

$$M^k = \begin{pmatrix} 1+k & -k \\ k & 1-k \end{pmatrix}.$$

Hence,

$$M^{26} = \begin{pmatrix} 27 & -26 \\ 26 & -25 \end{pmatrix}.$$

Step 4 : Summation

The $(1, 1)$-entry is

$$a = \sum_{k=1}^{26} (1 + k).$$

Thus,

$$a = 26 + \frac{26 \cdot 27}{2} = 26 + 351 = 377.$$

Hence, any statement claiming $a = 378$ is false.

Rishabh's Insight

Nilpotent decomposition transforms a seemingly difficult matrix power problem into elementary algebra.
This technique frequently appears in Olympiad-level linear algebra as well.

Pro Shortcut

Whenever a matrix has repeated eigenvalue 1, immediately test:

$$M - I.$$

If

$$(M - I)^2 = 0,$$

then

$$(I + N)^k = I + kN.$$

Final Answer

$$\boxed{\text{Options (A), (C), and (D)}}$$

Question 9 : Equivalence Relations and Set Partitions

Problem Statement

Let

$$S = \{1, 2, 3, \ldots, 10\}.$$

Let X denote the set of all equivalence relations R on S such that R contains exactly 42 ordered pairs.

Find

$$|X|.$$

Difficulty:
Theme: Equivalence Relations and Combinatorics
Target Time: 6 minutes

Core Mathematical Idea

Every equivalence relation corresponds uniquely to a partition of the set into equivalence classes.
If the class sizes are

$$n_1, n_2, \ldots, n_k,$$

then the total number of ordered pairs in the relation is

$$|R| = n_1^2 + n_2^2 + \cdots + n_k^2.$$

Step 1 : Conditions on Equivalence Classes

Suppose the equivalence classes have sizes

$$n_1, n_2, \ldots, n_k.$$

Since these classes partition the set S,

$$n_1 + n_2 + \cdots + n_k = 10.$$

Also, the number of ordered pairs in the equivalence relation is

$$n_1^2 + n_2^2 + \cdots + n_k^2 = 42.$$

We now determine all integer partitions satisfying both conditions.

Step 2 : Determining Possible Partitions

We seek partitions of 10 whose squares sum to 42.
Checking systematically:

$$6^2 + 2^2 + 1^2 + 1^2 = 36 + 4 + 1 + 1 = 42.$$

Hence,

$$(6, 2, 1, 1)$$

is a valid partition.

Also,

$$5^2 + 4^2 + 1^2 = 25 + 16 + 1 = 42.$$

Hence,

$$(5, 4, 1)$$

is also valid.

No other partition of 10 satisfies the required conditions.

Case 1 : Partition Type $(6, 2, 1, 1)$

We partition the set into equivalence classes of sizes

$$6,\ 2,\ 1,\ 1.$$

The number of such partitions is

$$\frac{10!}{6! \cdot 2! \cdot 1! \cdot 1!} \times \frac{1}{2!}.$$

The additional division by 2! arises because the two singleton classes are indistinguishable.

Therefore,

$$\frac{10!}{6! \cdot 2! \cdot 2!} = \frac{3628800}{720 \cdot 2 \cdot 2} = 1260.$$

Case 2 : Partition Type $(5, 4, 1)$

Now consider equivalence classes of sizes

$$5,\ 4,\ 1.$$

The number of such partitions is

$$\frac{10!}{5! \cdot 4! \cdot 1!}.$$

Hence,

$$\frac{3628800}{120 \cdot 24} = 1260.$$

Step 3 : Total Number of Equivalence Relations

Therefore,

$$|X| = 1260 + 1260 = 2520.$$

Rishabh's Insight

Equivalence relation problems are fundamentally partition problems.

The key observation is:

$$|R| = \sum n_i^2.$$

This transforms an abstract relation-counting problem into a structured integer partition problem combined with multinomial counting.

Exam Trap

A common mistake is treating equivalence classes as ordered groups.
Equivalence classes form an unordered partition of the set. Therefore, repeated class sizes introduce symmetry factors that must be divided out carefully.

Final Answer

$$\boxed{2520}$$

Question 10 : Continuity and Differentiability involving GIF

Problem Statement

Define
$$f(x) = (|x| + |x-1|)\sin x + [x \sin x]$$
on
$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
Let:
$$\alpha = \text{number of points of discontinuity,}$$
$$\beta = \text{number of points of non-differentiability.}$$
Find:
$$\alpha + \beta.$$

Difficulty:
Theme: Continuity, Differentiability and GIF
Target Time: 6 minutes

Core Mathematical Idea

The discontinuities arise entirely from the greatest integer function term.
Absolute value terms contribute only possible differentiability issues.

Step 1 : Discontinuity Analysis

Consider
$$g(x) = x \sin x.$$
Since:
$$x \text{ is odd}, \qquad \sin x \text{ is odd},$$
their product is even.
Further,
$$g(x) \geq 0$$
on
$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
Also,
$$g(0) = 0, \qquad g\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$
Hence,
$$0 \leq x \sin x < \frac{\pi}{2}.$$
Therefore, the only integer value crossed by $x \sin x$ is:
$$1.$$
Thus, discontinuities occur when:
$$x \sin x = 1.$$
On
$$\left(0, \frac{\pi}{2}\right),$$

observe:

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x > 0.$$

Hence, $x\sin x$ is strictly increasing on

$$\left(0, \frac{\pi}{2}\right).$$

Therefore, the equation

$$x\sin x = 1$$

has exactly one positive root.

By even symmetry, there is exactly one corresponding negative root.

Hence,

$$\alpha = 2.$$

Step 2 : Differentiability Analysis

The two discontinuity points are automatically non-differentiable.

Thus, we already have:

$$2$$

points contributing to β.

Now examine the absolute value terms.

At $x = 0$

Consider:

$$|x|\sin x.$$

Checking differentiability:

$$\lim_{h\to 0} \frac{|h|\sin h}{h}.$$

For $h > 0$,

$$\frac{|h|\sin h}{h} = \sin h \to 0.$$

For $h < 0$,

$$\frac{|h|\sin h}{h} = -\sin h \to 0.$$

Hence, derivative exists at $x = 0$.

Thus, no additional non-differentiable point occurs here.

At $x = 1$

The term

$$|x-1|\sin x$$

is non-differentiable at $x = 1$ because:

$$\sin 1 \neq 0.$$

Thus, one additional non-differentiable point occurs.

Therefore,

$$\beta = 2 + 1 = 3.$$

Step 3 : Final Computation

Hence,

$$\alpha + \beta = 2 + 3 = 5.$$

Rishabh's Insight

Expressions of the form

$$|x|u(x)$$

may still be differentiable at the origin if

$$u(0) = 0.$$

Never assume absolute value automatically destroys differentiability.

Exam Trap

Students often forget that discontinuity immediately implies non-differentiability.
Always count discontinuity points first before separately analyzing cusps and corners.

Final Answer

$$\boxed{5}$$

Question 11 : Distribution of Identical Objects under Constraints

Problem Statement

Ten identical Red pens and fourteen identical Blue pens are distributed among four persons such that each person receives exactly six pens.

Find the number of possible distributions.

Difficulty:
Theme: Integer Solutions and Inclusion-Exclusion
Target Time: 5 minutes

Core Mathematical Idea

The condition that each person receives exactly six pens converts the two-colour problem into a single constrained integer equation.

Step 1 : Formulation

Let:

$$r_i = \text{number of Red pens received by person } i.$$

Since each person receives exactly six pens,

$$b_i = 6 - r_i.$$

Total Red pens:

$$r_1 + r_2 + r_3 + r_4 = 10.$$

Since:

$$0 \le r_i \le 6,$$

the problem reduces to counting the number of non-negative integer solutions of:

$$r_1 + r_2 + r_3 + r_4 = 10$$

subject to

$$r_i \le 6.$$

Step 2 : Unrestricted Solutions

Ignoring the upper bound temporarily:

$$r_1 + r_2 + r_3 + r_4 = 10.$$

Number of non-negative integer solutions:

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286.$$

Step 3 : Inclusion-Exclusion

We subtract solutions where some:

$$r_i \geq 7.$$

Suppose:

$$r_1 \geq 7.$$

Write:

$$r_1' = r_1 - 7.$$

Then:

$$r_1' + r_2 + r_3 + r_4 = 3.$$

Number of solutions:

$$\binom{6}{3} = 20.$$

Similarly for each variable.
Hence, total invalid solutions:

$$4 \times 20 = 80.$$

No overlap cases are possible because:

$$7 + 7 > 10.$$

Therefore,

$$286 - 80 = 206.$$

Rishabh's Insight

Constraint-based distribution problems often simplify beautifully through inclusion-exclusion after converting colour constraints into a single-variable equation.

Whenever upper bounds appear in stars-and-bars problems, immediately think:

$$\text{Unrestricted Solutions} - \text{Violations}.$$

Final Answer

$$\boxed{206}$$

Question 12 : Telescoping Trigonometric Product

Problem Statement

Let

$$\alpha = \prod_{k=0}^{4}\left(1 - 2\cos\left(\frac{3^k\pi}{11}\right)\right).$$

Find:

$$5 - \alpha^2.$$

Difficulty:
Theme: Trigonometric Products and Telescoping
Target Time: 7 minutes

Core Mathematical Idea

Complicated trigonometric products frequently telescope after applying suitable angle identities.
The appearance of powers of 3 strongly suggests the triple-angle identity.

Step 1 : Key Identity

Using:

$$\sin 3x = \sin x(1 + 2\cos 2x),$$

put:

$$x = \frac{\theta}{2}.$$

Then,

$$\sin\frac{3\theta}{2} = \sin\frac{\theta}{2}\left(1 + 2\cos\theta\right).$$

Hence,

$$1 + 2\cos\theta = \frac{\sin(3\theta/2)}{\sin(\theta/2)}.$$

Therefore,

$$1 - 2\cos\theta = -\frac{\sin(3\theta/2)}{\sin(\theta/2)}.$$

Step 2 : Telescoping Product

Let:

$$\theta_k = \frac{3^k\pi}{11}.$$

Then:

$$\alpha = \prod_{k=0}^{4}\left(-\frac{\sin(3\theta_k/2)}{\sin(\theta_k/2)}\right).$$

Thus,

$$\alpha = (-1)^5\prod_{k=0}^{4}\frac{\sin\left(\frac{3^{k+1}\pi}{22}\right)}{\sin\left(\frac{3^k\pi}{22}\right)}.$$

All intermediate terms cancel telescopically.
Hence,

$$\alpha = -\frac{\sin\left(\frac{3^5\pi}{22}\right)}{\sin\left(\frac{\pi}{22}\right)}.$$

Since:

$$3^5 = 243,$$

we get:

$$\alpha = -\frac{\sin\left(\frac{243\pi}{22}\right)}{\sin\left(\frac{\pi}{22}\right)}.$$

Now:

$$\frac{243\pi}{22} = 11\pi + \frac{\pi}{22}.$$

Therefore,

$$\sin\left(11\pi + \frac{\pi}{22}\right) = -\sin\left(\frac{\pi}{22}\right).$$

Thus,

$$\alpha = -\frac{-\sin(\pi/22)}{\sin(\pi/22)} = 1.$$

Step 3 : Final Computation

Therefore,

$$5 - \alpha^2 = 5 - 1 = 4.$$

Rishabh's Insight

Whenever powers such as

$$3^k$$

appear inside trigonometric arguments, always test whether a telescoping structure emerges through multiple-angle identities.

Exam Trap

The most common mistake is attempting brute-force trigonometric simplification instead of searching for a telescoping identity.

Final Answer

$$\boxed{4}$$

Question 13 : Complex Numbers and Polynomial Transformations

Problem Statement

Let

$$\omega$$

be a complex cube root of unity satisfying

$$\omega^3 = 1, \qquad 1 + \omega + \omega^2 = 0.$$

Match the following expressions with their corresponding results.

Difficulty:
Theme: Roots of Unity and Polynomial Transformation
Target Time: 5 minutes

For cube roots of unity:

$$\omega^3 = 1.$$

Hence, powers of ω always reduce modulo 3.

(P)

The equation

$$x^2 + x + 1 = 0$$

has roots

$$\omega, \omega^2.$$

Now consider:

$$\frac{1}{(1+\omega)^{2026}}, \qquad \frac{1}{(1+\omega^2)^{2026}}.$$

Using:

$$1 + \omega = -\omega^2, \qquad 1 + \omega^2 = -\omega,$$

we get:

$$\frac{1}{(1+\omega)^{2026}} = \frac{1}{(-\omega^2)^{2026}}.$$

Since 2026 is even,

$$(-1)^{2026} = 1.$$

Also,

$$2026 \equiv 1 \pmod 3.$$

Thus,

$$\omega^{4052} = \omega^2.$$

Hence,

$$\frac{1}{\omega^2} = \omega.$$

Similarly,

$$\frac{1}{(1+\omega^2)^{2026}} = \omega^2.$$

Therefore, the transformed roots remain:

$$\omega, \omega^2.$$

Thus, the equation remains:

$$x^2 + x + 1 = 0.$$

$$\boxed{(P) \to (1)}$$

(Q)

Now power is:

$$2027.$$

Observe:

$$2027 \equiv 2 \pmod 3.$$

Then,

$$\frac{1}{(-\omega^2)^{2027}} = -\frac{1}{\omega^{4054}} = -\frac{1}{\omega} = -\omega^2.$$

Similarly,

$$\frac{1}{(-\omega)^{2027}} = -\omega.$$

Hence, roots become:

$$-\omega, -\omega^2.$$

Required equation:

$$x^2 - (\text{sum})x + (\text{product}) = 0.$$

Now,

$$(-\omega) + (-\omega^2) = -(\omega + \omega^2).$$

Using:

$$1 + \omega + \omega^2 = 0,$$

we get:

$$\omega + \omega^2 = -1.$$

Hence,

$$-(\omega + \omega^2) = 1.$$

Also,

$$(-\omega)(-\omega^2) = \omega^3 = 1.$$

Therefore,

$$x^2 - x + 1 = 0.$$

$$\boxed{(Q) \to (2)}$$

(R)

Roots of:

$$x^2 - x + 1 = 0$$

are:

$$-\omega, -\omega^2.$$

Now evaluate:

$$\frac{1}{(-\omega - 1)^{2026}} + \frac{1}{(-\omega^2 - 1)^{2026}}.$$

Using:

$$1 + \omega + \omega^2 = 0,$$

we get:

$$-(1 + \omega) = \omega^2, \qquad -(1 + \omega^2) = \omega.$$

Thus,

$$-\omega - 1 = \omega^2, \qquad -\omega^2 - 1 = \omega.$$

Therefore,

$$= \frac{1}{(\omega^2)^{2026}} + \frac{1}{\omega^{2026}}.$$

Since:

$$2026 \equiv 1 \pmod 3,$$

we get:

$$= \frac{1}{\omega^2} + \frac{1}{\omega} = \omega + \omega^2.$$

Again,

$$\omega + \omega^2 = -1.$$

Hence,

$$\boxed{(R) \longrightarrow (4)}$$

Rishabh's Insight

Roots of unity problems become extremely simple once powers are reduced modulo the order of the root.
For cube roots:

$$\omega^3 = 1$$

reduces every large exponent into one of:

$$1, \omega, \omega^2.$$

Exam Trap

Students often forget:

$$1 + \omega + \omega^2 = 0.$$

This identity is the backbone of almost every roots-of-unity simplification problem in JEE Advanced.

Final Matching

$$\boxed{(P) \longrightarrow (1), \quad (Q) \longrightarrow (2), \quad (R) \longrightarrow (4)}$$

Question 14 : Trigonometric Equations in Intervals

Problem Statement

Match the following trigonometric equations with the number of solutions in the specified interval.

Difficulty:
Theme: Trigonometric Equations
Target Time: 5 minutes

Core Mathematical Idea

For
$$0 < |t| < 1,$$
higher powers satisfy:
$$t^n < t^2.$$
Thus, equations of the form
$$\sin^m x + \cos^n x = 1$$
usually force boundary values such as:
$$0, \pm\frac{\pi}{2}, \pm\pi.$$

(P)

Solve:
$$\sin^6 x + \cos^4 x = 1.$$

Using:
$$1 - \sin^6 x = (1 - \sin^2 x)(1 + \sin^2 x + \sin^4 x),$$
we get:
$$\cos^4 x = \cos^2 x(1 + \sin^2 x + \sin^4 x).$$

Hence,
$$\cos^2 x = 1 + \sin^2 x + \sin^4 x$$
or
$$\cos x = 0.$$

Now:
$$\cos^2 x \leq 1,$$
while:
$$1 + \sin^2 x + \sin^4 x \geq 1.$$

Equality possible only when:
$$\sin x = 0.$$

Thus:
$$x = 0, \pm\pi.$$

Also,
$$\cos x = 0 \implies x = \pm\frac{\pi}{2}.$$

Total number of solutions:
$$5.$$

$$\boxed{(P) \to (5)}$$

(Q)

Solve:

$$\sin^2 x + \cos^6 x = 1.$$

Using:

$$1 - \sin^2 x = \cos^2 x,$$

we get:

$$\cos^6 x = \cos^2 x.$$

Thus,

$$\cos^2 x(\cos^4 x - 1) = 0.$$

Hence,

$$\cos x = 0$$

or

$$\cos^2 x = 1.$$

In:

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

solutions are:

$$-\frac{\pi}{2}, \quad 0, \quad \frac{\pi}{2}.$$

Total:

$$3.$$

$$\boxed{(Q) \to (3)}$$

Rishabh's Insight

Whenever even powers of sine and cosine appear, converting everything into either $\sin^2 x$ or $\cos^2 x$ often reveals hidden factorization structures.

Exam Trap

Never divide blindly by $\cos^2 x$ or $\sin^2 x$.
Always separately consider cases where these quantities may vanish.

Final Matching

$$\boxed{(P) \to (5), \qquad (Q) \to (3)}$$

Question 15 : Orthogonal Matrices and Vector Geometry

Problem Statement

Let

$$M = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \beta & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \gamma & \delta & \mu \end{pmatrix}$$

satisfy:

$$MM^T = I.$$

Match the following statements.

Difficulty:
Theme: Orthogonal Matrices
Target Time: 5 minutes

Core Mathematical Idea

If

$$MM^T = I,$$

then the rows and columns of M form orthonormal vectors.

(P)

First row:

$$\left(\alpha, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Since its norm is 1,

$$\alpha^2 + \frac{1}{2} + \frac{1}{2} = 1.$$

Hence,

$$\alpha = 0.$$

Now use third column normalization:

$$\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \mu^2 = 1.$$

Thus,

$$\frac{1}{2} + \frac{1}{3} + \mu^2 = 1.$$

Hence,

$$\mu^2 = \frac{1}{6}.$$

Now third row also has unit norm:

$$\gamma^2 + \delta^2 + \mu^2 = 1.$$

Therefore,

$$\gamma^2 + \delta^2 = 1 - \frac{1}{6} = \frac{5}{6}.$$

$$\boxed{(P) \longrightarrow (5)}$$

(R)

Let the row vectors be:

$$\vec{u}, \vec{v}, \vec{w}.$$

Now,

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det(M).$$

Since:

$$MM^T = I,$$

taking determinants gives:

$$(\det M)^2 = 1.$$

Hence,

$$\det M = \pm 1.$$

Therefore,

$$\left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right| = 1.$$

$$\boxed{(R) \longrightarrow (2)}$$

Rishabh's Insight

Orthogonal matrices preserve:

length, angle, volume.

That is why their determinant magnitude is always:

1.

Exam Trap

Students often use only row orthogonality and forget that columns are also orthonormal in an orthogonal matrix.

Final Matching

$$\boxed{(P) \longrightarrow (5), \qquad (R) \longrightarrow (2)}$$

Question 16 : Conic Sections and Tangents

Problem Statement

Match the following conic-section statements with their corresponding results.

Difficulty:
Theme: Conic Sections
Target Time: 6 minutes

(P)

Circle centered at:

$$(1, 2)$$

touches the line:

$$3x + 4y - 1 = 0.$$

Radius equals perpendicular distance from center to line:

$$R = \frac{|3(1) + 4(2) - 1|}{\sqrt{3^2 + 4^2}} = \frac{10}{5} = 2.$$

Hence equation:

$$(x - 1)^2 + (y - 2)^2 = 4.$$

Checking:

$$(3, 2)$$

satisfies the equation.

Therefore,

$$\boxed{(P) \longrightarrow (3)}$$

(Q)

Parabola:

$$y^2 = 8x.$$

Tangent in slope form:

$$y = mx + \frac{2}{m}.$$

Distance from origin to tangent is:

$$\sqrt{2}.$$

Thus,

$$\frac{|2/m|}{\sqrt{1 + m^2}} = \sqrt{2}.$$

Squaring:

$$\frac{4}{m^2(1 + m^2)} = 2.$$

Hence,

$$m^4 + m^2 - 2 = 0.$$

Factorizing:

$$(m^2 - 1)(m^2 + 2) = 0.$$

Thus,

$$m^2 = 1.$$

Taking positive slope:

$$m = 1.$$

Required tangent:

$$y = x + 2.$$

Checking:

$$(7, 9)$$

lies on it.

Hence,

$$\boxed{(Q) \longrightarrow (2)}$$

(S)

Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Given:

$$c = 5, \qquad \frac{a}{e} = \frac{16}{5}.$$

Since:

$$ae = c,$$

we get:

$$(ae)\left(\frac{a}{e}\right) = a^2.$$

Thus,

$$a^2 = 16.$$

Hence,

$$a = 4.$$

Now:

$$b^2 = c^2 - a^2 = 25 - 16 = 9.$$

Equation becomes:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

Checking point:

$$(8, 3\sqrt{3}) :$$
$$\frac{64}{16} - \frac{27}{9} = 4 - 3 = 1.$$

Hence,

$$\boxed{(S) \longrightarrow (5)}$$

Rishabh's Insight

Slope-form tangents are often the fastest route in advanced conic problems.
For parabola:

$$y^2 = 4ax,$$

always remember:

$$y = mx + \frac{a}{m}.$$

Exam Trap

In conic problems, parameter relations such as:

$$c^2 = a^2 + b^2$$

for hyperbolas and

$$c^2 = a^2 - b^2$$

for ellipses are frequently confused.

Final Matching

$$\boxed{(P) \longrightarrow (3), \qquad (Q) \longrightarrow (2), \qquad (S) \longrightarrow (5)}$$