International Baccalaureate Diploma Programme Mathematics Analysis and Approaches Higher Level

Paper 3 Elite Edition

Unlock 7-Scorer Potential

Exclusive IB Exam-Style Solved Problems Based on May 2021 TZ2
Practice Problems | Rishabh's Insight | May 2025 Edition

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Introduction

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Problem 1 [Total Marks: 31]

Consider the polynomial function $g_n(x) = x^n(b-x)^n$, where $b > 0$ is a real constant and n is a positive integer. For parts (a) and (b), assume $b = 2$.

(a) For $g_1(x) = x(2-x)$, sketch the graph of $y = g_1(x)$, indicating all intercepts with the axes and the coordinates of any local maximum or minimum points. [3 marks]

(b) For $g_n(x) = x^n(2-x)^n$ with $n > 1$, use a graphing calculator to investigate the graph of $y = g_n(x)$ for:

- Odd values $n = 3$ and $n = 5$
- Even values $n = 2$ and $n = 4$

Complete the table below based on your observations:

	Number of local maximum points	Number of local minimum points	Number of points of inflection with zero gradient
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

[6 marks]

(c) For $g_n(x) = x^n(b-x)^n$ with $b > 0$ and $n > 1$, show that the derivative is $g_n'(x) = nx^{n-1}(b-2x)(b-x)^{n-1}$. [3 marks]

(d) List the three values of x where $g_n'(x) = 0$. [2 marks]

(e) Prove that the point $\left(\frac{b}{2}, g_n\left(\frac{b}{2}\right)\right)$ lies above the x-axis. [2 marks]

(f) Show that $g_n'\left(\frac{b}{4}\right) > 0$ for all positive integers n. [2 marks]

(g) Using the result from part (f) and the sign of $g_n'(-1)$, demonstrate that the point $(0, 0)$ is:

(i) A local minimum for even $n > 1$ and $b > 0$. [3 marks]

(ii) A point of inflection with zero gradient for odd $n > 1$ and $b > 0$. [3 marks]

(h) For the function $y = x^n(b - x)^n - k$ with n a positive integer, $b > 0$, and k real, determine the conditions on n and k such that $x^n(b - x)^n = k$ has four real solutions for x. [5 marks]

Solution to Problem 1

Solution to Problem 1(a)

We are given $g_1(x) = x(2-x)$ with $b = 2$. $g_1(x) = 2x - x^2$.

Intercepts with the axes:

- y-intercept: Set $x = 0 \implies g_1(0) = 0(2-0) = 0$. Point $(0, 0)$.
- x-intercepts: Set $g_1(x) = 0 \implies x(2-x) = 0 \implies x = 0$ or $x = 2$. Points $(0, 0)$ and $(2, 0)$.

Local maximum or minimum points:

Find the derivative: $g_1'(x) = \frac{d}{dx}(2x - x^2) = 2 - 2x$.

Set $g_1'(x) = 0 \implies 2 - 2x = 0 \implies 2x = 2 \implies x = 1$.

The value of the function at $x = 1$ is $g_1(1) = 1(2-1) = 1$. So, $(1, 1)$ is a stationary point.

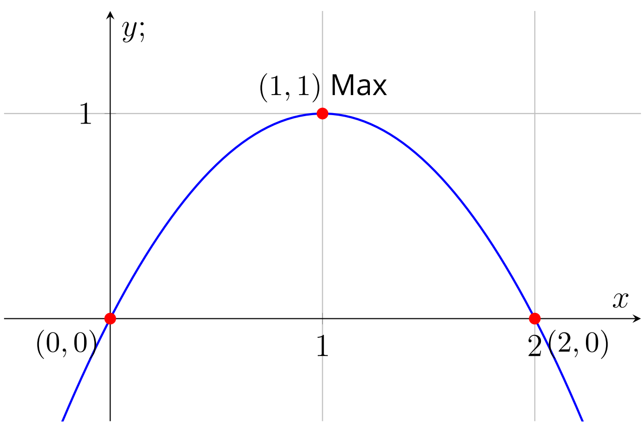
To classify this point, find the second derivative: $g_1''(x) = \frac{d}{dx}(2 - 2x) = -2$.

Since $g_1''(1) = -2 < 0$, the point $(1, 1)$ is a local maximum.

The graph is a parabola opening downwards.

$$\boxed{\text{Intercepts: } (0,0), (2,0). \text{ Local maximum: } (1,1)}$$

Sketch of $g_1(x) = x(2-x)$:



Explanation: The graph of $g_1(x) = x(2-x)$ is a downward-opening parabola with

x-intercepts at $(0, 0)$ and $(2, 0)$, and a local maximum at $(1, 1)$.

Solution to Problem 1(b)

For $g_n(x) = x^n(2-x)^n = (x(2-x))^n$.

For odd values $n = 3, 5$ *(and* $n > 1$*)*:

The function $g_n(x)$ has a local maximum at $x = 1$, where $g_n(1) = (1(2-1))^n = 1^n = 1$.

At $x = 0$ and $x = 2$, $g_n(x) = 0$. $g_n'(x) = nx^{n-1}(2-x)^{n-1}(2-2x)$.

Since n is odd, $n-1$ is even. For $x \approx 0$ ($x \neq 0$), $x^{n-1} > 0$. $(2-x)^{n-1} > 0$. $(2-2x)$ determines sign of $g_n'(x)$ near $x = 0$ and $x = 2$.

Near $x = 0$, $2 - 2x > 0$. So $g_n'(x) > 0$ for x near 0 ($x \neq 0$). No sign change. Point of inflection with zero gradient.

Near $x = 2$, $2 - 2x < 0$. So $g_n'(x) < 0$ for x near 2 ($x \neq 2$). No sign change. Point of inflection with zero gradient.

Thus, 1 local maximum, 0 local minimum points, 2 points of inflection with zero gradient (at $x = 0$ and $x = 2$).

For even values $n = 2, 4$ *(and* $n > 1$*)*:

The function $g_n(x)$ has a local maximum at $x = 1$, $g_n(1) = 1$.

At $x = 0$ and $x = 2$, $g_n(x) = 0$. Since n is even, $g_n(x) \geq 0$ for all x. Thus, $x = 0$ and $x = 2$ are local minimum points.

There are no points of inflection with zero gradient (as these are minima).

Thus, 1 local maximum, 2 local minimum points, 0 points of inflection with zero gradient.

Completed table:

	Number of local maximum points	Number of local minimum points	Number of points of inflection with zero gradient
$n = 3$ and $n = 5$	1	0	2
$n = 2$ and $n = 4$	1	2	0

Odd n: 1 max, 0 min, 2 POI z.g. Even n: 1 max, 2 min, 0 POI z.g.

Solution to Problem 1(c)

Given $g_n(x) = x^n(b-x)^n = (x(b-x))^n = (bx-x^2)^n$.

Using the chain rule: Let $u(x) = bx - x^2$. Then $g_n(x) = (u(x))^n$.

$g_n'(x) = n(u(x))^{n-1} \cdot u'(x)$.

$u'(x) = \frac{d}{dx}(bx - x^2) = b - 2x$.

So, $g_n'(x) = n(bx-x^2)^{n-1}(b-2x)$.

Substituting $bx - x^2 = x(b-x)$: $g_n'(x) = n(x(b-x))^{n-1}(b-2x) = nx^{n-1}(b-x)^{n-1}(b-2x)$.

$$\boxed{g_n'(x) = nx^{n-1}(b-2x)(b-x)^{n-1}}$$

Solution to Problem 1(d)

To find values of x where $g_n'(x) = 0$, we solve $nx^{n-1}(b-x)^{n-1}(b-2x) = 0$.
Since $n > 1$ (given in part c, also n is positive integer implies $n \geq 1$), $n \neq 0$.
The equation holds if any of the variable factors are zero:

- $x^{n-1} = 0 \implies x = 0$ (since $n - 1 \geq 1$ because $n > 1$ means $n \geq 2$).
- $(b-x)^{n-1} = 0 \implies b - x = 0 \implies x = b$.
- $b - 2x = 0 \implies 2x = b \implies x = \frac{b}{2}$.

The three values of x are $0, \frac{b}{2}, b$.

$$\boxed{x = 0, \quad x = \frac{b}{2}, \quad x = b}$$

Solution to Problem 1(e)

We need to evaluate $g_n\left(\frac{b}{2}\right)$.

$g_n(x) = x^n(b-x)^n$.

$g_n\left(\frac{b}{2}\right) = \left(\frac{b}{2}\right)^n\left(b-\frac{b}{2}\right)^n = \left(\frac{b}{2}\right)^n\left(\frac{b}{2}\right)^n = \left(\left(\frac{b}{2}\right)^2\right)^n = \left(\frac{b^2}{4}\right)^n$.

Since $b>0$, $b^2>0$. Therefore $\frac{b^2}{4}>0$. As n is a positive integer, $\left(\frac{b^2}{4}\right)^n>0$.

So, $g_n\left(\frac{b}{2}\right)>0$, which means the point $\left(\frac{b}{2}, g_n\left(\frac{b}{2}\right)\right)$ lies above the x-axis.

$$\boxed{g_n\left(\frac{b}{2}\right) = \left(\frac{b^2}{4}\right)^n > 0, \text{ so the point lies above the } x\text{-axis}}$$

Solution to Problem 1(f)

We need to evaluate $g_n'\left(\frac{b}{4}\right)$.

Using $g_n'(x) = nx^{n-1}(b-x)^{n-1}(b-2x)$:

$$\begin{aligned} g_n'\left(\frac{b}{4}\right) &= n\left(\frac{b}{4}\right)^{n-1}\left(b-\frac{b}{4}\right)^{n-1}\left(b-2\left(\frac{b}{4}\right)\right) \\ &= n\left(\frac{b}{4}\right)^{n-1}\left(\frac{3b}{4}\right)^{n-1}\left(b-\frac{b}{2}\right) \\ &= n\left(\frac{b}{4}\cdot\frac{3b}{4}\right)^{n-1}\left(\frac{b}{2}\right) \\ &= n\left(\frac{3b^2}{16}\right)^{n-1}\left(\frac{b}{2}\right) \end{aligned}$$

Since n is a positive integer, $n > 0$.

Since $b > 0$, then $\frac{3b^2}{16} > 0$ and $\frac{b}{2} > 0$.

If $n = 1$, $g_1'\left(\frac{b}{4}\right) = 1\left(\frac{3b^2}{16}\right)^0\left(\frac{b}{2}\right) = \frac{b}{2} > 0$.

If $n > 1$, then $n-1 \geq 1$. $\left(\frac{3b^2}{16}\right)^{n-1} > 0$.

Therefore, $g_n'\left(\frac{b}{4}\right) = n \underbrace{\left(\frac{3b^2}{16}\right)^{n-1}}_{\geq 0 \text{ (actually } >0 \text{ if } n>1 \text{ or } =1 \text{ if } n=1)} \underbrace{\left(\frac{b}{2}\right)}_{>0} > 0.$

So, $g_n'\left(\frac{b}{4}\right) > 0$ for all positive integers n.

$$\boxed{g_n'\left(\frac{b}{4}\right) = n\left(\frac{3b^2}{16}\right)^{n-1}\left(\frac{b}{2}\right) > 0}$$

Solution to Problem 1(g)(i)

We want to show $(0,0)$ is a local minimum for even $n > 1$ and $b > 0$.

We know $g_n(0) = 0$. We know $g_n'(0) = 0$ from part (d). We need to examine the sign of $g_n'(x)$ for x near 0.

$g_n'(-1) = n(-1)^{n-1}(b - (-1))^{n-1}(b - 2(-1)) = n(-1)^{n-1}(b + 1)^{n-1}(b + 2)$.

Since n is even and $n > 1$, $n - 1$ is odd. So $(-1)^{n-1} = -1$.

$g_n'(-1) = n(-1)(b + 1)^{n-1}(b + 2)$. Since $n > 0, b > 0$, $(b + 1)^{n-1} > 0, (b + 2) > 0$.

Thus, $g_n'(-1) < 0$.

We are given $g_n'\left(\frac{b}{4}\right) > 0$ from part (f). Since $b > 0$, $-1 < 0 < \frac{b}{4}$.

The derivative changes sign from negative (at x like -1, and thus for $x \in (-\delta, 0)$ for some $\delta > 0$) to positive (at $x = \frac{b}{4}$, and thus for $x \in (0, \epsilon)$ for some $\epsilon > 0$) at $x = 0$.

Therefore, $(0,0)$ is a local minimum for even $n > 1$.

$\boxed{g_n'(-1) < 0 \text{ and } g_n'(b/4) > 0\text{. Sign change } - \to + \text{ at } x = 0\text{. So, local minimum.}}$

Solution to Problem 1(g)(ii)

We want to show $(0,0)$ is a point of inflection with zero gradient for odd $n > 1$ and $b > 0$.

We know $g_n(0) = 0$ and $g_n'(0) = 0$.

Consider $g_n'(-1) = n(-1)^{n-1}(b+1)^{n-1}(b+2)$. Since n is odd and $n > 1$, $n-1$ is even. So $(-1)^{n-1} = 1$.

$g_n'(-1) = n(1)(b+1)^{n-1}(b+2)$. Since $n > 0, b > 0$, then $g_n'(-1) > 0$.

We are given $g_n'\left(\frac{b}{4}\right) > 0$.

The derivative $g_n'(x)$ is positive for $x \in (-1,0)$ (more accurately, for $x \in (-\delta, 0)$) and for $x \in (0, \frac{b}{4})$ (more accurately, for $x \in (0, \epsilon)$).

So $g_n'(x)$ does not change sign at $x = 0$ (it is positive on both sides of 0, close to 0).

Since $g_n'(0) = 0$ and $g_n'(x)$ does not change sign at $x = 0$, it is a point of inflection with zero gradient.

$$\boxed{g_n'(-1) > 0 \text{ and } g_n'(b/4) > 0\text{. No sign change } + \to + \text{ at } x = 0\text{. So, POI z.g.}}$$

Solution to Problem 1(h)

We are looking for conditions on n and k such that $x^n(b-x)^n = k$ has four real solutions for x.
This is $g_n(x) = k$. We need the horizontal line $y = k$ to intersect the graph of $y = g_n(x)$ four times.

- If n is odd ($n > 1$): From (g)(ii), $(0, 0)$ is a POI z.g. By symmetry, $(b, 0)$ is also a POI z.g. $\left(\frac{b}{2}, g_n\left(\frac{b}{2}\right)\right)$ is a local maximum. The function increases from $x = 0$ to $x = b/2$ and then decreases from $x = b/2$ to $x = b$.
  For $x < 0$, $g_n(x) = (x(b-x))^n < 0$. For $x > b$, $g_n(x) < 0$.
  A horizontal line $y = k$ can intersect at most twice if $0 \le k \le g_n(b/2)$. So, for odd n, not four real solutions.

- If n is even ($n > 1$): From (g)(i), $(0, 0)$ is a local minimum. By symmetry, $(b, 0)$ is also a local minimum.
  The point $\left(\frac{b}{2}, g_n\left(\frac{b}{2}\right)\right) = \left(\frac{b}{2}, \left(\frac{b^2}{4}\right)^n\right)$ is a local maximum.
  The graph starts high ($g_n(x) \to \infty$ as $x \to \pm\infty$), goes down to local minimum $(0, 0)$, up to local maximum $\left(\frac{b}{2}, \left(\frac{b^2}{4}\right)^n\right)$, down to local minimum $(b, 0)$, and then up high.
  For $y = k$ to intersect four times: k must be greater than the value at the local minima $(0, 0)$ and $(b, 0)$, and less than the value at the local maximum.
  So, $0 < k < \left(\frac{b^2}{4}\right)^n$. Also, n must be an even positive integer, and $n > 1$ (as $n = 1$ gives a parabola with at most 2 solutions).

$$\boxed{n \text{ is an even integer, } n \ge 2, \text{ and } 0 < k < \left(\frac{b^2}{4}\right)^n}$$

Alternative Solutions to Problem 1

Alternative Solution to Problem 1(c)

Using the product rule for $g_n(x) = x^n(b-x)^n$:

Let $U(x) = x^n$ and $V(x) = (b-x)^n$. Then $U'(x) = nx^{n-1}$ and $V'(x) = n(b-x)^{n-1}(-1)$.

$g_n'(x) = U'(x)V(x) + U(x)V'(x) = nx^{n-1}(b-x)^n + x^n(-n(b-x)^{n-1}) = nx^{n-1}(b-x)^n - nx^n(b-x)^{n-1}$.

Factor out common terms $nx^{n-1}(b-x)^{n-1}$:

$g_n'(x) = nx^{n-1}(b-x)^{n-1}[(b-x)-x] = nx^{n-1}(b-x)^{n-1}(b-2x)$. This matches the previous result.

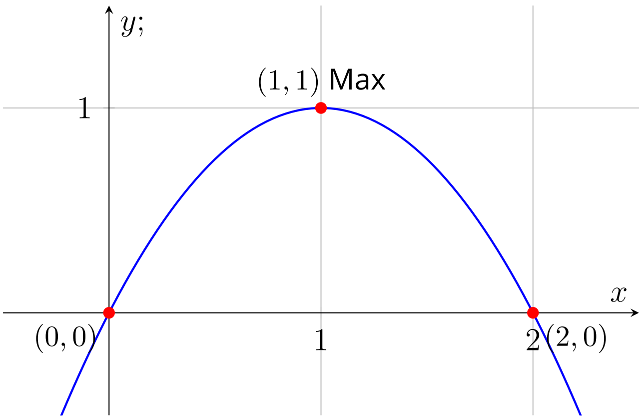
Strategy to Analyze Polynomial Functions

1. **Identify Function Type**: Symmetrical polynomial $g_n(x) = (x(b-x))^n$. Degree $2n$.
2. **Find Domain**: $\mathbb{R}$ for polynomials.
3. **Find Intercepts**: $y = 0 \implies x(b-x) = 0 \implies x = 0$ or $x = b$. $g_n(0) = 0$.
4. **Check Symmetry**: $g_n(x) = (x(b-x))^n$. Let $X = x - b/2$. Then $x = X + b/2$, $b - x = b/2 - X$. So $x(b-x) = (b/2)^2 - X^2$. $g_n(x) = ((b/2)^2 - (x - b/2)^2)^n$. This is symmetric about $x = \frac{b}{2}$.
5. **First Derivative**: Find critical points ($g_n'(x) = 0$). These are $x = 0, b/2, b$. Classify them by analyzing the sign of $g_n'(x)$ around them, considering the parity of n.
6. **Behavior at** $x = 0, b$: Depends on parity of n. If n is even ($n > 1$), $g_n(x) \geq 0$, so $(0,0)$ and $(b,0)$ are local minima. If n is odd ($n > 1$), $(0,0)$ and $(b,0)$ are points of inflection with zero gradient.
7. **Behavior at** $x = b/2$: $g_n(b/2) = (b^2/4)^n > 0$. This is a local maximum as $x(b-x)$ is maximized at $x = b/2$.
8. **Sketch Graph**: Plot key points. Consider end behavior: For even n, $x(b-x) \sim -x^2$ for large $|x|$, so $(x(b-x))^n \sim (-x^2)^n = x^{2n} \to \infty$ as $x \to \pm\infty$. For odd n, $(x(b-x))^n \sim (-x^2)^n = -x^{2n}$ (this is incorrect logic for sign). Correct end behavior: $x^n(b-x)^n = x^n(-1)^n(x-b)^n \approx (-1)^n x^{2n}$. If n is even, $g_n(x) \to \infty$ as $x \to \pm\infty$. If n is odd, $g_n(x) \to -\infty$ as $x \to \pm\infty$.
9. **Label Clearly**: Mark intercepts, maxima, minima, points of inflection on sketches.

Visualization

Sketch for Problem 1(a)

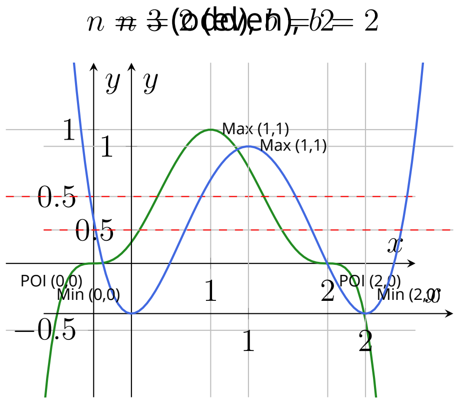
$g_1(x) = x(2 - x)$



Explanation: The graph of $g_1(x) = x(2 - x)$ is a downward-opening parabola with x-intercepts at $(0, 0)$ and $(2, 0)$, and a local maximum at $(1, 1)$.

Sketches for Problem 1(h) understanding

Example sketches for $g_n(x) = x^n(b - x)^n$ illustrating why n must be even for 4 solutions to $g_n(x) = k$. Let $b = 2$. Max value is $((2^2/4))^n = 1^n = 1$.



Explanation: For $n = 3$ (odd), $g_n(x) \to -\infty$ as $x \to \pm\infty$. A line $y = k = 0.5$ intersects twice. For $n = 2$ (even), $g_n(x) \to \infty$ as $x \to \pm\infty$. $y = k = 0.5$ intersects four times.

Marking Criteria

Marking Criteria for Problem 1

Polynomial Functions (Total 31 marks):

- **Part (a) Sketch** [3 marks]
  - **A1** for correct x-intercepts $(0, 0)$ and $(2, 0)$.
  - **A1** for correct local maximum $(1, 1)$.
  - **A1** for correct shape (downward parabola).
- **Part (b) Table from GDC** [6 marks]
  - For $n = 3, 5$: **A1** for 1 local max. **A1** for 0 local min. **A1** for 2 POI with zero gradient.
  - For $n = 2, 4$: **A1** for 1 local max. **A1** for 2 local min. **A1** for 0 POI with zero gradient.
- **Part (c) Derivative** [3 marks]
  - **M1** for correct application of chain rule (or product rule).
  - **A1** for intermediate step e.g. $n(x(b - x))^{n-1}(b - 2x)$.
  - **A1** for final expression $nx^{n-1}(b - 2x)(b - x)^{n-1}$. AG.
- **Part (d) Roots of derivative** [2 marks]
  - **A1** for $x = 0, x = b$. **A1** for $x = b/2$. (Award A1 for two correct, A2 for all three).
- **Part (e) Point above x-axis** [2 marks]
  - **M1** for evaluating $g_n(b/2) = (b/2)^n(b - b/2)^n$.
  - **A1** for $g_n(b/2) = (b^2/4)^n$ and stating it's > 0 as $b > 0$. AG.
- **Part (f) Sign of $g'_n(b/4)$** [2 marks]
  - **M1** for substituting $x = b/4$ into $g'_n(x)$.
  - **A1** for simplifying to $n(3b^2/16)^{n-1}(b/2)$ and stating it's > 0. AG.

Continued...

Polynomial Functions (Total 31 marks):

- **Part (g)(i) Local minimum for even** n [3 marks]
  - **M1** for evaluating $g_n'(-1)$ and finding its sign (negative for even n).
  - **R1** for noting $g_n'(0) = 0$.
  - **A1** for concluding local minimum due to sign change of $g_n'(x)$ from $-$ to $+$ around $x = 0$ (using $g_n'(-1) < 0$ and $g_n'(b/4) > 0$).
- **Part (g)(ii) Inflection point for odd** n [3 marks]
  - **M1** for evaluating $g_n'(-1)$ and finding its sign (positive for odd n).
  - **R1** for noting $g_n'(0) = 0$.
  - **A1** for concluding POI with zero gradient due to $g_n'(x)$ not changing sign (stays $+$) around $x = 0$.
- **Part (h) Conditions for four real solutions** [5 marks]
  - **R1** for stating n must be even.
  - **R1** for identifying $n \geq 2$ (or $n > 1$).
  - **M1** for considering the range of k relative to local extrema values $(0$ and $(b^2/4)^n)$.
  - **A1** for $k > 0$.
  - **A1** for $k < (b^2/4)^n$.

Error Analysis: Common Mistakes and Fixes for Polynomial Functions

Mistake	Explanation	How to Fix It
Derivative errors	Misapplying chain rule or product rule for $g_n'(x)$, especially with exponents $n-1$.	Write $g_n(x) = (x(b-x))^n$. Let $u = x(b-x)$. Then $g_n(x) = u^n$. $g_n'(x) = nu^{n-1}u'$. Calculate u' correctly. Or carefully apply product rule to $x^n(b-x)^n$.
Missing critical points	Forgetting one of the roots of $g_n'(x) = 0$, often $x = b/2$.	Ensure all factors x^{n-1}, $(b-x)^{n-1}$, $(b-2x)$ are considered when setting $g_n'(x) = 0$. Valid for $n > 1$.
Incorrect table (part b)	Misinterpreting GDC output, especially for points of inflection with zero gradient vs. local minima/maxima.	Verify that for POI with zero gradient, $g_n'(x) = 0$ but $g_n'(x)$ does not change sign. For local min/max, $g_n'(x) = 0$ and $g_n'(x)$ changes sign. Check roots $x = 0, 2$ carefully based on n's parity.
Sign analysis errors (part g)	Mistakes in determining the sign of $g_n'(-1)$ or $g_n'(b/4)$, especially with $(-1)^{n-1}$.	Carefully consider if $n-1$ is odd or even based on whether n is odd or even. $b+1, b+2$ are positive since $b > 0$.
Conditions for 4 roots (part h)	Not realizing n must be even, or incorrect range for k. Assuming behavior of $n = 1$ or $n = 2$ applies to all n.	Sketch or visualize $g_n(x)$ for odd and even n. Four roots require "W" shape (even n) and k to be between the y-value of minima and maximum. $n \geq 2$ for even n.

Mistake	Explanation	How to Fix It
POI classification	Confusing criteria for point of inflection with zero gradient. In part (b), GDC might not explicitly state "POI with zero gradient".	These occur when $g_n'(x) = 0$ and $g_n''(x) = 0$, and $g_n''(x)$ changes sign (or $g_n'(x)$ doesn't change sign around x_0 but $g_n'(x_0) = 0$). For n odd, $x = 0$ and $x = b$ are such points.

Key Takeaways

- The structure of $g_n(x) = (x(b-x))^n$ means its properties are related to the simpler quadratic $x(b-x)$.
- The parity of n (odd or even) is crucial for determining the nature of stationary points at $x = 0$ and $x = b$.
  - If n is even ($n > 1$), $g_n(x) \geq 0$, so $(0, 0)$ and $(b, 0)$ are local minima.
  - If n is odd ($n > 1$), $g_n(x)$ can be negative. $(0, 0)$ and $(b, 0)$ are points of inflection with zero gradient.
- The point $x = b/2$ corresponds to the axis of symmetry of $x(b-x)$, and thus is always a local maximum for $g_n(x)$ when $x \in (0, b)$. (More precisely, $x(b-x)$ is maximized, so $g_n(x)$ will have a local max there too).
- The number of solutions to $g_n(x) = k$ depends on the shape of $g_n(x)$ (determined by n) and the value of k relative to local extrema.
- Sign analysis of the first derivative $g_n'(x)$ around a critical point x_0 (where $g_n'(x_0) = 0$) is a robust way to classify it.

Rishabh's Insights - Shortcuts & Tricks

- **Symmetry is Key**: Recognize that $g_n(x) = x^n(b-x)^n$ is symmetric about $x = b/2$. This means if you analyze behavior at $x = 0$, the behavior at $x = b$ is analogous. E.g., if $(0,0)$ is a local min, $(b,0)$ is also a local min for even n.
- **Base Function** $h(x) = x(b-x)$: Think of $g_n(x) = (h(x))^n$. The critical points of $g_n(x)$ will include critical points of $h(x)$ (i.e. $x = b/2$) and roots of $h(x)$ (i.e. $x = 0, b$).
- **Parity Power**: For terms like x^{n-1} or $(b-x)^{n-1}$ in $g_n'(x)$, their sign depends critically on the parity of $n-1$.
  If n is even, $n-1$ is odd: x^{n-1} changes sign at $x = 0$.
  If n is odd, $n-1$ is even: x^{n-1} is positive for $x \neq 0$ (if $n-1 > 0$).
- **GDC for Table (Part b)**: Use the GDC to quickly observe patterns for specified n. Ensure your GDC window shows all relevant features, especially near $x = 0$ and $x = 2$.
- **Check Smallest n for AG parts**: For "Show that..." questions like part (c) or (f), if you're stuck on the general algebra, test with $n = 2$ (or $n = 1$ if applicable) to see if your specific case matches the target. For $g_n'(x)$, $n = 1$ yields $g_1'(x) = 1 \cdot x^0(b-2x)(b-x)^0 = b - 2x$, which is correct for $g_1(x) = bx - x^2$.
- **Part (h) - Visual Reasoning**: For the number of roots of $g_n(x) = k$, a quick mental sketch based on the classification of critical points (minima at $0, b$ and max at $b/2$ for even n) is very effective.

Basic Foundational Theory

- **Polynomials**: A function of the form $P(x) = a_k x^k + \cdots + a_1 x + a_0$. The degree of $g_n(x) = x^n(b-x)^n$ is $2n$.
- **Intercepts**: x-intercepts are found by setting $y = 0$. y-intercept is found by setting $x = 0$.
- **Stationary Points (Critical Points)**: Points where the first derivative $f'(x)$ is zero or undefined. For polynomials, $f'(x) = 0$.
- **Classifying Stationary Points**:
  - **First Derivative Test**: Check sign of $f'(x)$ on either side of the critical point.
  - **Second Derivative Test**: If $f'(c) = 0$: If $f''(c) > 0 \implies$ local minimum. If $f''(c) < 0 \implies$ local maximum. If $f''(c) = 0 \implies$ test is inconclusive.
- **Points of Inflection (POI)**: Points where concavity changes. Occur when $f''(x) = 0$ or is undefined, AND $f''(x)$ changes sign around that point. A POI with zero gradient has $f'(x) = 0$ and $f''(x) = 0$ (and $f''(x)$ changes sign).
- **Symmetry**: A function $f(x)$ is symmetric about $x = c$ if $f(c-h) = f(c+h)$. $g_n(x) = x^n(b-x)^n$ is symmetric about $x = b/2$.

Practice Problems 1

Practice Problem 1: Polynomial Graph

Let $f(x) = x(4 - x)$. Sketch the graph of $y = f(x)$, indicating all intercepts with the axes and the coordinates of any local maximum or minimum points. [3 marks]

Solution to Practice Problem 1

$f(x) = 4x - x^2$. *Intercepts*: y-int: $f(0) = 0 \implies (0, 0)$. x-int: $x(4 - x) = 0 \implies x = 0, x = 4 \implies (0, 0), (4, 0)$.

Local max/min: $f'(x) = 4 - 2x$. $f'(x) = 0 \implies x = 2$. $f(2) = 2(4 - 2) = 4$. Point $(2, 4)$. $f''(x) = -2 < 0 \implies (2, 4)$ is a local maximum.

$$\boxed{\text{Intercepts: } (0,0), (4,0);\ \text{Local maximum: } (2,4)}$$

Practice Problem 2: Critical Points

Find the x-coordinates of the critical points of $h(x) = x^2(1 - x)^2$. [3 marks]

Solution to Practice Problem 2

This is $g_n(x)$ with $n = 2$ and $b = 1$. So $g_n'(x) = nx^{n-1}(b - x)^{n-1}(b - 2x)$.

$h'(x) = 2x^{2-1}(1-x)^{2-1}(1-2x) = 2x(1-x)(1-2x)$. Set $h'(x) = 0 \implies 2x(1-x)(1-2x) = 0$.

Critical points at $x = 0$, $x = 1$, $x = 1/2$.

$$\boxed{x = 0, x = 1/2, x = 1}$$

Further Problems 1

Further Problem 1: Polynomial Analysis

Consider $f_n(x) = x^n(c-x)^n$ where $c < 0$ and n is an even positive integer ($n \geq 2$). Find the x-coordinates of any local maxima and local minima. [5 marks]

Solution to Further Problem 1

This is $g_n(x)$ with $b = c$. $f_n'(x) = nx^{n-1}(c-x)^{n-1}(c-2x)$.
Critical points are $x = 0, x = c, x = c/2$. Since n is even ($n \geq 2$), $f_n(x) = (x(c-x))^n \geq 0$.
$f_n(0) = 0$ and $f_n(c) = 0$. These are the lowest possible values, so $x = 0$ and $x = c$ are local minima.
The point $x = c/2$ is between c and 0 (since $c < 0$, $c/2$ is also negative, and $c < c/2 < 0$).
$f_n(c/2) = ((c/2)(c - c/2))^n = ((c/2)(c/2))^n = (c^2/4)^n$. Since $c \neq 0$, $c^2/4 > 0$, so $f_n(c/2) > 0$.
Since $x = 0, c$ are local minima (value 0), and $f_n(c/2) > 0$, $x = c/2$ must be a local maximum.

$$\boxed{\text{Local minima at } x = 0, x = c. \text{ Local maximum at } x = c/2.}$$

Further Problem 2: Root Conditions

For $h_n(x) = x^n(2-x)^n$, where n is an odd integer $n \geq 3$. Determine the number of distinct real solutions to $h_n(x) = k$ for $k \in \mathbb{R}$. Consider different ranges of k. [4 marks]

Solution to Further Problem 2

For n odd ($n \geq 3$):
$(0,0)$ and $(2,0)$ are POI z.g. $(1, h_n(1)) = (1,1)$ is a local maximum.
End behavior: As $x \to \pm\infty$, $x(2-x) \sim -x^2 \to -\infty$. So $h_n(x) = (x(2-x))^n \to (-\infty)^n = -\infty$ (since n is odd).
Graph comes from $-\infty$, up to POI $(0,0)$, continues up to local max $(1,1)$, down to POI $(2,0)$, continues down to $-\infty$.

Number of solutions for $h_n(x) = k$:

- If $k > 1$: No solutions.
- If $k = 1$: One solution (at $x = 1$).
- If $0 < k < 1$: Two distinct solutions.
- If $k = 0$: Two distinct solutions (at $x = 0, x = 2$).
- If $k < 0$: Two distinct solutions.

$$\begin{cases} 0 \text{ solutions} & \text{if } k > 1 \\ 1 \text{ solution} & \text{if } k = 1 \\ 2 \text{ solutions} & \text{if } k \leq 0 \text{ or } 0 < k < 1 \implies k < 1 \end{cases} \quad \text{Refined: } \begin{cases} 0 & k > 1 \\ 1 & k = 1 \\ 2 & k < 1 \end{cases}$$

Problem 2 [Total Marks: 24]

This question is about the n-th roots of unity. Let $z \in \mathbb{C}$. The n-th roots of $z^n = 1$ can be written as $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. These roots can be represented as points $P_0, P_1, P_2, \ldots, P_{n-1}$ respectively, on a circle of radius 1 centred at the origin $O(0,0)$ in an Argand diagram. Let P_0P_k denote the distance from point P_0 to point P_k.

(a) Consider the case $n = 3$. The roots are $1, \omega, \omega^2$.

(i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2 marks]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2 marks]

(b) For $n = 3$, show that $P_0P_1 \cdot P_0P_2 = 3$. [4 marks]

(c) For $n = 4$, the roots are $1, \omega, \omega^2, \omega^3$. By factorizing $z^4 - 1$ or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [3 marks]

(d) For $n = 4$, show that $P_0P_1 \cdot P_0P_2 \cdot P_0P_3 = 4$. [4 marks]

(e) For a general integer $n \geq 2$, suggest a value for the product $P_0P_1 \cdot P_0P_2 \cdot \ldots \cdot P_0P_{n-1}$. [1 mark]

(f) (i) Write down expressions for the distances P_0P_2 and P_0P_3 in terms of ω. [2 marks]

(ii) Write down an expression for the distance P_0P_{n-1} in terms of n and ω. [1 mark]

(g) Consider the identity $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$. The roots of $z^n - 1 = 0$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

(i) Express $z^{n-1} + z^{n-2} + \ldots + z + 1$ as a product of linear factors in terms of z and ω. [2 marks]

(ii) By substituting an appropriate value for z, prove your suggested value in part (e). [3 marks]

Solution to Problem 2

Solution to Problem 2(a)(i)

Expand the left hand side (LHS):

LHS = $(\omega - 1)(\omega^2 + \omega + 1) = \omega(\omega^2 + \omega + 1) - 1(\omega^2 + \omega + 1) = (\omega^3 + \omega^2 + \omega) - (\omega^2 + \omega + 1)$
$= \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1 = \omega^3 - 1.$

This is equal to the right hand side (RHS).

$$\boxed{(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1}$$

Solution to Problem 2(a)(ii)

For $n = 3$, $\omega = e^{\frac{2\pi i}{3}}$ is a root of $z^3 = 1$. So, $\omega^3 = 1$.

Therefore, $\omega^3 - 1 = 0$.

From part (a)(i), $(\omega - 1)(\omega^2 + \omega + 1) = 0$.

Since $\omega = e^{\frac{2\pi i}{3}} \neq 1$, the factor $(\omega - 1) \neq 0$.

Thus, the other factor must be zero: $\omega^2 + \omega + 1 = 0$.

$$\boxed{\omega^2 + \omega + 1 = 0}$$

Solution to Problem 2(b)

The points are P_0 representing $z_0 = 1$, P_1 representing $z_1 = \omega$, and P_2 representing $z_2 = \omega^2$.

The distance P_0P_k is $|z_0 - z_k| = |1 - z_k|$. $P_0P_1 = |1 - \omega|$.

$P_0P_2 = |1 - \omega^2|$.

The product is $P_0P_1 \cdot P_0P_2 = |1 - \omega||1 - \omega^2| = |(1 - \omega)(1 - \omega^2)|$.

$(1 - \omega)(1 - \omega^2) = 1 - \omega^2 - \omega + \omega^3$.

Since $\omega^3 = 1$ and from (a)(ii) $\omega^2 + \omega = -1$:

$1 - (\omega^2 + \omega) + \omega^3 = 1 - (-1) + 1 = 1 + 1 + 1 = 3$.

So, $P_0P_1 \cdot P_0P_2 = |3| = 3$.

$$\boxed{P_0P_1 \cdot P_0P_2 = 3}$$

Solution to Problem 2(c)

For $n = 4$, $\omega = e^{\frac{2\pi i}{4}} = e^{\frac{\pi i}{2}} = i$.
The roots of $z^4 - 1 = 0$ are $1, \omega, \omega^2, \omega^3$.
We have $z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$.
Since ω is a root of $z^4 - 1 = 0$ and $\omega \neq 1$, ω must be a root of $z^3 + z^2 + z + 1 = 0$.
Therefore, $\omega^3 + \omega^2 + \omega + 1 = 0$.

$$\boxed{\omega^3 + \omega^2 + \omega + 1 = 0}$$

Solution to Problem 2(d)

For $n = 4$, the points are $P_0(1), P_1(\omega), P_2(\omega^2), P_3(\omega^3)$.

The product of distances is $P_0P_1 \cdot P_0P_2 \cdot P_0P_3 = |1-\omega||1-\omega^2||1-\omega^3|$.

This is equal to $|(1-\omega)(1-\omega^2)(1-\omega^3)|$.

From part (c), the polynomial $Q(z) = z^3 + z^2 + z + 1$ has roots $\omega, \omega^2, \omega^3$.

So, $z^3 + z^2 + z + 1 = (z-\omega)(z-\omega^2)(z-\omega^3)$ (since leading coefficient is 1).

Substitute $z = 1$ into this polynomial:

$1^3 + 1^2 + 1 + 1 = (1-\omega)(1-\omega^2)(1-\omega^3)$.

So, $(1-\omega)(1-\omega^2)(1-\omega^3) = 1 + 1 + 1 + 1 = 4$.

The product of distances is $|4| = 4$.

$$\boxed{P_0P_1 \cdot P_0P_2 \cdot P_0P_3 = 4}$$

Solution to Problem 2(e)

For $n = 3$, the product is 3. For $n = 4$, the product is 4.
Based on this pattern, for a general integer $n \geq 2$, the product $P_0P_1 \cdot P_0P_2 \cdot \ldots \cdot P_0P_{n-1}$ is n.

$$\boxed{n}$$

Solution to Problem 2(f)(i)

P_0 represents $z_0 = 1$. P_k represents $z_k = \omega^k$.

The distance $P_0P_k = |1 - \omega^k|$.

$P_0P_2 = |1 - \omega^2|$.

$P_0P_3 = |1 - \omega^3|$.

$$\boxed{P_0P_2 = |1 - \omega^2|, \quad P_0P_3 = |1 - \omega^3|}$$

Solution to Problem 2(f)(ii)

The point P_{n-1} represents the root ω^{n-1}. The distance $P_0P_{n-1} = |1 - \omega^{n-1}|$.

$$\boxed{P_0P_{n-1} = |1 - \omega^{n-1}|}$$

Solution to Problem 2(g)(i)

The polynomial $z^n - 1$ has roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

So, $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$.

We are given $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$.

Comparing the two factorizations by dividing by $(z-1)$ (for $z \neq 1$):

$z^{n-1} + z^{n-2} + \ldots + z + 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$.

$$\boxed{z^{n-1} + z^{n-2} + \ldots + z + 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})}$$

Solution to Problem 2(g)(ii)

The product of distances is $P_0P_1 \cdot P_0P_2 \cdot \ldots \cdot P_0P_{n-1} = |1-\omega| \cdot |1-\omega^2| \cdot \ldots \cdot |1-\omega^{n-1}|$.
This can be written as $|(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})|$.
Using the result from part (g)(i):
$z^{n-1} + z^{n-2} + \ldots + z + 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$.
Substitute $z = 1$ into this identity:
LHS: $1^{n-1} + 1^{n-2} + \ldots + 1 + 1$. There are n terms, each equal to 1. So, LHS = n.
RHS: $(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})$.
So, $n = (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})$.
Taking the modulus of both sides:
$|n| = |(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})|$. Since $n \geq 2$, n is a positive real number, so $|n| = n$.
$n = |1-\omega| \cdot |1-\omega^2| \cdot \ldots \cdot |1-\omega^{n-1}|$.
This is exactly the product $P_0P_1 \cdot P_0P_2 \cdot \ldots \cdot P_0P_{n-1}$.
Therefore, the product is n. This proves the suggestion from part (e).

$$\boxed{P_0P_1 \cdot P_0P_2 \cdot \ldots \cdot P_0P_{n-1} = n}$$

Alternative Solutions to Problem 2

Alternative Solution to Problem 2(b) (Geometric/Trigonometric)

For $n = 3$, roots are $P_0(1)$, $P_1(e^{i2\pi/3})$, $P_2(e^{i4\pi/3})$.

$P_0 = (1, 0)$. $P_1 = (\cos(2\pi/3), \sin(2\pi/3)) = (-1/2, \sqrt{3}/2)$.

$P_2 = (\cos(4\pi/3), \sin(4\pi/3)) = (-1/2, -\sqrt{3}/2)$.

$(P_0P_1)^2 = (1 - (-1/2))^2 + (0 - \sqrt{3}/2)^2 = (3/2)^2 + (-\sqrt{3}/2)^2 = 9/4 + 3/4 = 12/4 = 3$.

So $P_0P_1 = \sqrt{3}$.

$(P_0P_2)^2 = (1 - (-1/2))^2 + (0 - (-\sqrt{3}/2))^2 = (3/2)^2 + (\sqrt{3}/2)^2 = 9/4 + 3/4 = 12/4 = 3$.

So $P_0P_2 = \sqrt{3}$.

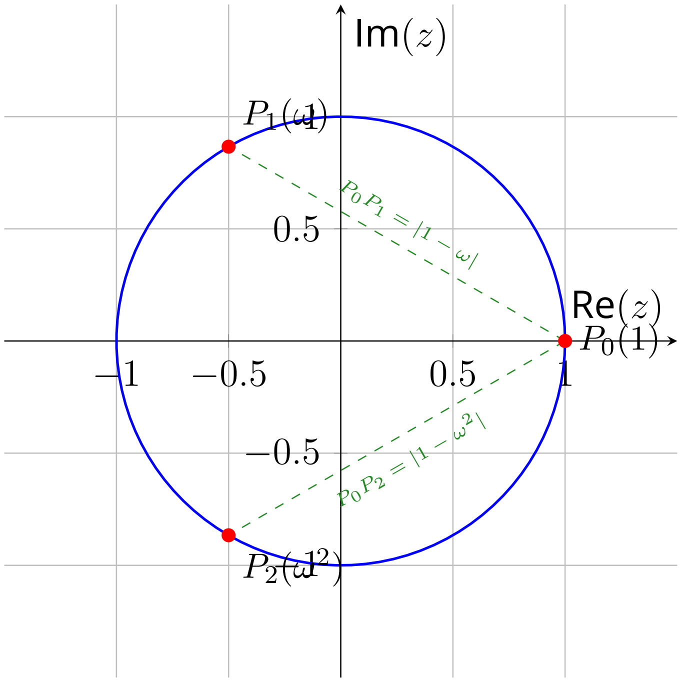
$P_0P_1 \cdot P_0P_2 = \sqrt{3} \cdot \sqrt{3} = 3$.

Strategy to Analyze Complex Roots of Unity

1. **Definition of ω**: Understand $\omega = e^{2\pi i/n}$ and that $\omega^n = 1$. The roots are $1, \omega, \ldots, \omega^{n-1}$.
2. **Polynomial Factorization**: Key identities are $z^n - 1 = \prod_{k=0}^{n-1}(z - \omega^k)$ and $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + 1)$.
3. **Roots of $Q(z)$**: The roots of $Q(z) = z^{n-1} + z^{n-2} + \cdots + 1$ are $\omega, \omega^2, \ldots, \omega^{n-1}$.
4. **Geometric Interpretation**: Roots are vertices of a regular n-gon inscribed in the unit circle in the Argand diagram. P_0 is at $(1, 0)$.
5. **Distance Formula**: The distance between points representing complex numbers z_a and z_b is $|z_a - z_b|$. In this problem, $P_0P_k = |1 - \omega^k|$.
6. **Product of Distances**: The problem revolves around evaluating $\prod_{k=1}^{n-1} |1 - \omega^k|$. This is achieved by substituting $z = 1$ into $Q(z) = \prod_{k=1}^{n-1}(z - \omega^k)$ and taking the modulus.
7. **Value of $Q(1)$**: $Q(1) = 1^{n-1} + 1^{n-2} + \cdots + 1 = n$ (sum of n ones).

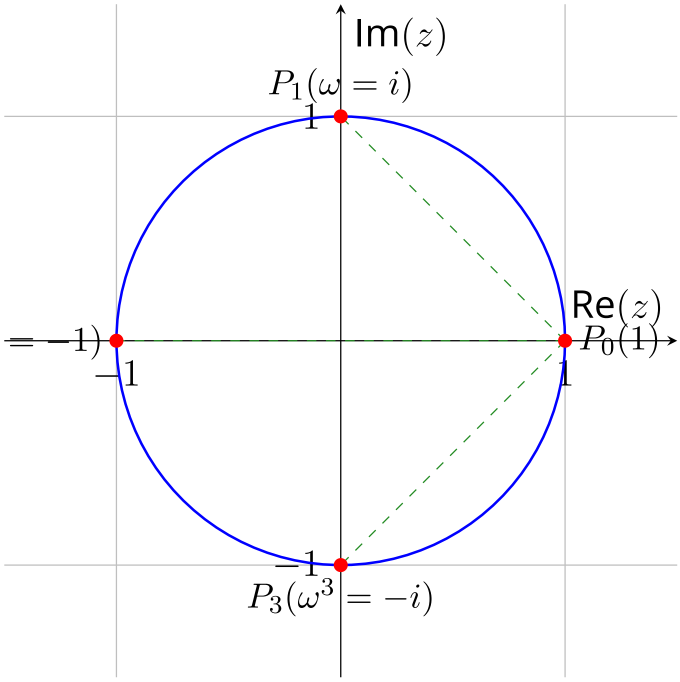
Visualization

Roots of Unity for $n = 3$



Explanation: The three cube roots of unity on the unit circle. P_0 is at 1. P_1 is $\omega = e^{i2\pi/3}$. P_2 is $\omega^2 = e^{i4\pi/3}$. The dashed lines show the distances P_0P_1 and P_0P_2.

Roots of Unity for $n = 4$



Explanation: The four fourth roots of unity on the unit circle: $1, i, -1, -i$. Dashed lines indicate distances from P_0.

Marking Criteria

Marking Criteria for Problem 2

Complex Roots of Unity (Total 24 marks):

- **Part (a)(i) Expansion** [2 marks]
  - **M1** for attempting to expand LHS. **A1** for correct expansion to $\omega^3 - 1$. AG.
- **Part (a)(ii) Deduction** [2 marks]
  - **R1** for $\omega^3 - 1 = 0$. **R1** for $\omega - 1 \neq 0 \implies \omega^2 + \omega + 1 = 0$. AG.
- **Part (b) Product for $n = 3$** [4 marks]
  - **M1** for $P_0P_1 = |1-\omega|$, $P_0P_2 = |1-\omega^2|$. **M1** for product $|(1-\omega)(1-\omega^2)|$.
  - **M1** for expanding $(1 - \omega)(1 - \omega^2) = 1 - \omega - \omega^2 + \omega^3$. **A1** for using $\omega^3 = 1, \omega + \omega^2 = -1$ to get 3. AG.
- **Part (c) Deduction for $n = 4$** [3 marks]
  - **M1** for $z^4 - 1 = (z - 1)(z^3 + z^2 + z + 1)$. **R1** for $\omega \neq 1$ is a root of $z^4 - 1 = 0$.
  - **A1** for deducing ω is a root of $z^3 + z^2 + z + 1 = 0$. AG.
- **Part (d) Product for $n = 4$** [4 marks]
  - **M1** for product as $|(1-\omega)(1-\omega^2)(1-\omega^3)|$. **R1** for linking to $z^3 + z^2 + z + 1$.
  - **M1** for substituting $z = 1$ into $z^3 + z^2 + z + 1$. **A1** for result 4. AG.
- **Part (e) Suggestion for general n** [1 mark]
  - **A1** for n.
- **Part (f)(i) Expressions for distances** [2 marks]
  - **A1** for $P_0P_2 = |1 - \omega^2|$. **A1** for $P_0P_3 = |1 - \omega^3|$.
- **Part (f)(ii) Expression for P_0P_{n-1}** [1 mark]
  - **A1** for $P_0P_{n-1} = |1 - \omega^{n-1}|$.

Marking Criteria for Problem 2 - Continued...

Complex Roots of Unity (Total 24 marks): Continued...

- **Part (g)(i) Product of linear factors** [2 marks]
  - **M1** for using roots $\omega, \ldots, \omega^{n-1}$ for $z^{n-1} + \cdots + 1$.
  - **A1** for $(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$. AG.
- **Part (g)(ii) Proof of suggestion** [3 marks]
  - **M1** for substituting $z = 1$ into $z^{n-1} + \cdots + 1 = \prod(z - \omega^k)$.
  - **A1** for LHS sum is n. **A1** for RHS product leads to n after modulus. AG.

Error Analysis: Common Mistakes and Fixes for Complex Roots

Mistake	Explanation	How to Fix It
Algebraic expansion	Errors in expanding products like $(\omega-1)(\omega^2+\omega+1)$ or $(1-\omega)(1-\omega^2)$.	Distribute terms carefully, one by one. Keep track of signs. Use $\omega^n = 1$ at the appropriate step.
Deduction logic	In (a)(ii) or (c), stating $\omega^2+\omega+1=0$ without properly justifying why $\omega-1\neq 0$.	Explicitly state that $\omega = e^{2\pi i/n}$ (for $n \geq 2$) is not equal to 1, so the factor $(\omega-1)$ is non-zero.
Modulus errors	Calculating $\|(1-\omega)(1-\omega^2)\|$ as $\|1-\omega\|\|1-\omega^2\|$ but then forgetting the modulus for the final result if the product $(1-\omega)(1-\omega^2)$ is already real and positive. Or, errors in calculating specific moduli like $\|1-i\|$.	Remember $\|AB\| = \|A\|\|B\|$. The product of distances is inherently positive. If $(1-\omega)(1-\omega^2)\ldots$ evaluates to a real number X, the product of moduli is $\|X\|$. For $\|a+bi\| = \sqrt{a^2+b^2}$.
Factorization confusion	Incorrectly stating the roots of $z^{n-1}+\cdots+1=0$, or how it relates to $z^n-1=0$.	Roots of $z^n-1=0$ are $1,\omega,\ldots,\omega^{n-1}$. Since $z^n-1=(z-1)(z^{n-1}+\cdots+1)$, the roots of $z^{n-1}+\cdots+1=0$ must be $\omega,\ldots,\omega^{n-1}$.
Substitution step (g)(ii)	Substituting $z=1$ into $z^n-1=\prod(z-\omega^k)$ instead of into $z^{n-1}+\cdots+1=\prod_{k=1}^{n-1}(z-\omega^k)$. Or algebraic error in summing $1+1+\cdots+1$.	The expression for the product of distances $\|(1-\omega)\ldots(1-\omega^{n-1})\|$ matches the factored form of $z^{n-1}+\cdots+1$ evaluated at $z=1$. The sum $1^{n-1}+\cdots+1$ has n terms.

Key Takeaways

- The n-th roots of unity, $z_k = \omega^k = e^{i2\pi k/n}$ for $k = 0, 1, \ldots, n-1$, are fundamental in complex analysis. $z_0 = 1$.
- The identity $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ is crucial.
- The roots of $Q(z) = z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$ are $\omega, \omega^2, \ldots, \omega^{n-1}$.
- Thus, $Q(z)$ can be factorized as $Q(z) = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$.
- The distance from P_0 (representing 1) to P_k (representing ω^k) is $|1-\omega^k|$.
- The product of these distances (excluding P_0 to itself) is $\prod_{k=1}^{n-1} |1-\omega^k|$.
- This product is $|Q(1)|$. Since $Q(1) = 1^{n-1} + \ldots + 1 = n \times 1 = n$. So the product is n.
- This result has a geometric interpretation: the product of the lengths of the chords from one vertex of a regular n-gon inscribed in a unit circle to all other vertices is n.

Rishabh's Insights - Shortcuts & Tricks

- **Sum of Roots**: For $n \geq 2$, $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. This is often useful. For example, in 2(a)(ii), $\omega^2 + \omega + 1 = 0$ is directly this sum for $n = 3$. In 2(c), for $n = 4$, $\omega^3 + \omega^2 + \omega + 1 = 0$ (where $\omega = i$) is part of this sum excluding the '1' term.

- **Geometric Intuition**: Visualizing the roots on the Argand diagram as vertices of a regular n-gon helps. For $n = 4$, roots are $1, i, -1, -i$, forming a square. This makes distance calculations like $|1 - i|$ or $|1 - (-1)|$ very intuitive.

- **Symmetry of Distances**: $|1-\omega^k| = |1-\omega^{n-k}|$. This is because $\omega^{n-k} = \omega^n \omega^{-k} = \omega^{-k} = \overline{\omega^k}$ (conjugate if ω is on unit circle). So $|1 - \omega^k| = |1 - \overline{\omega^k}|$. Geometrically, the chord P_0P_k has the same length as P_0P_{n-k}.

- **Polynomial Root Product Property**: For a polynomial $a_m z^m + \cdots + a_0 = a_m \prod(z - r_i)$, if we need $\prod(c - r_i)$, it's $(a_m c^m + \cdots + a_0)/a_m$. Here, $Q(z) = z^{n-1} + \cdots + 1$, so the leading coefficient $a_{n-1} = 1$. The product $\prod_{k=1}^{n-1}(1 - \omega^k)$ is $Q(1)/1 = Q(1)$.

- **Small n Verification**: If unsure about a general step, quickly verify with $n = 3$ or $n = 4$. For example, in (g)(i), for $n = 3$, $z^2 + z + 1 = (z - \omega)(z - \omega^2)$, which is known from basic quadratic theory if roots are ω, ω^2.

Basic Foundational Theory

- **Roots of Unity**: Solutions to $z^n = 1$ are $z_k = e^{i\frac{2\pi k}{n}}$ for $k = 0, 1, \ldots, n-1$. Let $\omega = e^{i\frac{2\pi}{n}}$, then roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.
- **Properties of** ω: $\omega^n = 1$. $|\omega| = 1$. $\overline{\omega} = \omega^{-1} = \omega^{n-1}$ (for ω on unit circle).
- **Sum of n-th Roots of Unity**: $\sum_{k=0}^{n-1} \omega^k = 1 + \omega + \ldots + \omega^{n-1} = 0$ for $n \geq 2$. (This is from sum of a geometric series $\frac{\omega^n - 1}{\omega - 1} = \frac{1-1}{\omega - 1} = 0$ for $\omega \neq 1$).
- **Polynomial Factorization**: If a polynomial $P(z)$ has roots $r_1, \ldots, r_m$ and leading coefficient a_m, then $P(z) = a_m(z - r_1) \cdots (z - r_m)$.
- **Modulus of a Complex Number**: For $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$.
- **Properties of Modulus**: $|z_1 z_2| = |z_1||z_2|$. $|z_1/z_2| = |z_1|/|z_2|$.
- **Distance in Argand Diagram**: Distance between points representing z_1 and z_2 is $|z_1 - z_2|$.

Practice Problems 2

Practice Problem 1: $n = 5$ sum of roots

For $n = 5$, let $\omega = e^{2\pi i/5}$. Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. [2 marks]

Solution to Practice Problem 1

The sum $S = 1 + \omega + \omega^2 + \omega^3 + \omega^4$ is a geometric series with first term $a = 1$, ratio $r = \omega$, and $N = 5$ terms.
$S = \frac{a(r^N-1)}{r-1} = \frac{1(\omega^5-1)}{\omega-1}$.
Since $\omega = e^{2\pi i/5}$, $\omega^5 = (e^{2\pi i/5})^5 = e^{2\pi i} = 1$.
So $S = \frac{1-1}{\omega-1} = \frac{0}{\omega-1}$. Since $\omega \neq 1$ (as $n = 5 \geq 2$), $\omega - 1 \neq 0$.
Thus $S = 0$.

$$\boxed{1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0}$$

Practice Problem 2: Distance for $n = 6$

For $n = 6$, find the distance P_0P_3. P_0 represents 1, P_3 represents ω^3. [2 marks]

Solution to Practice Problem 2

For $n = 6$, $\omega = e^{2\pi i/6} = e^{\pi i/3}$. P_3 represents $\omega^3 = (e^{\pi i/3})^3 = e^{\pi i}$.
$e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$. P_0 represents 1.
The distance $P_0P_3 = |1 - \omega^3| = |1 - (-1)| = |1 + 1| = |2| = 2$.

$$\boxed{P_0P_3 = 2}$$

Further Problems 2

Further Problem 1: Product of all chord lengths from P_0

Let $P_0, P_1, \ldots, P_{n-1}$ be the vertices of a regular n-gon inscribed in a circle of radius R centered at the origin. If P_0 is at $(R, 0)$ (i.e., represents the complex number R), find the product of the lengths of the chords from P_0 to all other vertices $P_1, \ldots, P_{n-1}$. [4 marks]

Solution to Further Problem 1

The vertices are $z_k = R\omega^k = Re^{i2\pi k/n}$ for $k = 0, \ldots, n-1$.
P_0 represents $z_0 = R\omega^0 = R$.
The distance from P_0 to P_k is $|z_0 - z_k| = |R - R\omega^k| = |R(1 - \omega^k)| = R|1 - \omega^k|$ (since $R > 0$).
The product of the lengths of the chords from P_0 to all other vertices $P_1, \ldots, P_{n-1}$ is:
Product $= \prod_{k=1}^{n-1}(R|1 - \omega^k|) = R^{n-1} \prod_{k=1}^{n-1} |1 - \omega^k|$.
From Problem 2(g)(ii), we established that $\prod_{k=1}^{n-1} |1 - \omega^k| = n$.
So the product is $R^{n-1} \cdot n$.

$$\boxed{nR^{n-1}}$$

Further Problem 2: Sum of squares of chord lengths from P_0

For the n-th roots of unity $1, \omega, \ldots, \omega^{n-1}$ (points $P_0, \ldots, P_{n-1}$ on the unit circle), find the sum of the squares of the distances from P_0 to all other points: $S = \sum_{k=1}^{n-1}(P_0P_k)^2$. [6 marks]

Solution to Further Problem 2

$(P_0P_k)^2 = |1-\omega^k|^2$.
$|1-\omega^k|^2 = (1-\omega^k)(1-\overline{\omega^k})$. Since ω is on the unit circle, $\overline{\omega^k} = \omega^{-k}$.
So, $|1-\omega^k|^2 = (1-\omega^k)(1-\omega^{-k}) = 1-\omega^{-k}-\omega^k+\omega^k\omega^{-k} = 1-\omega^{-k}-\omega^k+1 = 2-(\omega^k+\omega^{-k})$.
The sum $S = \sum_{k=1}^{n-1}[2-(\omega^k+\omega^{-k})]$.
$S = \sum_{k=1}^{n-1} 2 - \sum_{k=1}^{n-1}(\omega^k+\omega^{-k})$.
$\sum_{k=1}^{n-1} 2 = 2(n-1)$.
We know $1+\omega+\omega^2+\ldots+\omega^{n-1} = 0$. So $\sum_{k=1}^{n-1}\omega^k = -1$.
Also, $\omega^{-k} = \omega^{n-k}$. The set $\{\omega^{-1}, \omega^{-2}, \ldots, \omega^{-(n-1)}\}$ is the same as $\{\omega^{n-1}, \omega^{n-2}, \ldots, \omega^1\}$ (just reordered).
So $\sum_{k=1}^{n-1}\omega^{-k} = \sum_{j=1}^{n-1}\omega^j = -1$.
Therefore, $\sum_{k=1}^{n-1}(\omega^k+\omega^{-k}) = \sum_{k=1}^{n-1}\omega^k + \sum_{k=1}^{n-1}\omega^{-k} = (-1)+(-1) = -2$.
So, $S = 2(n-1)-(-2) = 2n-2+2 = 2n$.

$$\boxed{\sum_{k=1}^{n-1}(P_0P_k)^2 = 2n}$$

Conclusion: Your Path to Mathematical Mastery

This guide has provided you with a powerful toolset for tackling IB Math AA HL Paper 3 challenges. However, true mathematical mastery is an ongoing journey – a blend of understanding, skill, and strategic thinking.

All solutions and commentary are original work by Rishabh Kumar, Mathematics Elevate Academy.

Key Takeaways for Exam Success:

- **Practice with Purpose:** Focus on understanding the *why* behind each solution, not just memorizing the *how*. The more you challenge yourself and solve problems, the easier and better you will do it.
- **Embrace Your Mistakes:** Every mistake is an opportunity to learn. Analyze what worked and what you can improve next time.
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