International Baccalaureate Diploma Programme Mathematics Analysis and Approaches Higher Level

Paper 3 2022 TZ1 Elite Edition

Unlock 7-Scorer Potential

Exclusive IB Exam-Style Solved Problems May 2022 TZ1 Practice Problems | Rishabh's Insight | May 2025 Edition

Mathematics Elevate Academy

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Introduction

Unlock your mathematical potential with **Mathematics Elevate Academy's** exclusive solved problem set IB Math AA HL Paper 3 May 2022 TZ1, crafted for ambitious IB DP Mathematics AA HL students.

This collection provides a *rigorous and enriching* preparation experience tailored for the current syllabus (2021 examinations onward).

This guide empowers you to:

- **Master Elite-Level Challenges:** Enhance your depth of understanding with questions that go beyond the textbook.
- **Understand the IB Marking Scheme:** Step-by-step examiner-style solutions show how to score full marks.
- **Avoid Hidden Pitfalls:** Efficient strategies and structured thinking save time under pressure.
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Contents

Problem 1 [Total Marks: 27]

Polygonal numbers are integers represented as dots forming regular polygons, such as triangular, square, or pentagonal numbers. For an r-sided regular polygon ($r \in \mathbb{Z}^+, r \geq 3$), the nth polygonal number is:

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \quad n \in \mathbb{Z}^+$$

(a) For triangular numbers ($r = 3$):

(i) Verify that $P_3(n) = \frac{n(n+1)}{2}$. [2 marks]

(ii) Determine which triangular number is 351. [2 marks]

(b) For triangular numbers:

(i) Show that $P_3(n) + P_3(n+1) = (n+1)^2$. [2 marks]

(ii) Describe what part (b)(i) implies about consecutive triangular numbers. [1 mark]

(iii) For $n = 4$, sketch a diagram illustrating part (b)(ii). [1 mark]

(c) Prove that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3 marks]

(d) The nth pentagonal number is given by the series $P_5(n) = 1+4+7+\ldots+(3n-2)$. Show that $P_5(n) = \frac{n(3n-1)}{2}$. [3 marks]

(e) Find the smallest positive integer greater than 1 that is both a triangular and a pentagonal number. [5 marks]

(f) Prove using mathematical induction that $P_r(n) = \frac{(r-2)n^2-(r-4)n}{2}$ for the series $\sum_{m=1}^{n}[1 + (m-1)(r-2)]$. [4 marks]

Solution to Problem 1

Solution to Problem 1(a)(i)

For triangular numbers, $r = 3$:

$$P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

$$\boxed{P_3(n) = \frac{n(n+1)}{2}}$$

Solution to Problem 1(a)(ii)

Solve:

$$\frac{n(n+1)}{2} = 351 \implies n^2 + n - 702 = 0$$

Quadratic formula:

$$n = \frac{-1 \pm \sqrt{1 + 4 \cdot 702}}{2} = \frac{-1 \pm \sqrt{2809}}{2} = \frac{-1 \pm 53}{2}$$

$$n = 26 \quad \text{or} \quad n = -27$$

Since $n \in \mathbb{Z}^+$, $n = 26$. Verify: $P_3(26) = \frac{26 \cdot 27}{2} = 351$.

$$\boxed{\text{26th triangular number, } n = 26}$$

Solution to Problem 1(b)(i)

$$P_3(n) = \frac{n(n+1)}{2}, \quad P_3(n+1) = \frac{(n+1)(n+2)}{2}$$

$$P_3(n)+P_3(n+1) = \frac{n(n+1) + (n+1)(n+2)}{2} = \frac{(n+1)(n+n+2)}{2} = \frac{(n+1)(2n+2)}{2} = (n+1)^2$$

$$\boxed{P_3(n) + P_3(n+1) = (n+1)^2}$$

Solution to Problem 1(b)(ii)

The sum of the nth and $(n+1)$th triangular numbers equals the $(n+1)$th square number.

$$\boxed{\text{Sum is the } (n+1)\text{th square number}}$$

Solution to Problem 1(b)(iii)

For $n=4$: - $P_3(4) = \frac{4\cdot 5}{2} = 10$. - $P_3(5) = \frac{5\cdot 6}{2} = 15$. - $P_3(4) + P_3(5) = 10 + 15 = 25 = 5^2$.

The diagram shows a 5×5 grid of dots, with a triangular arrangement of 10 dots and an L-shaped region of 15 dots forming a square.

$$\boxed{\text{Diagram shows triangles summing to a square}}$$

Solution to Problem 1(c)

$$8P_3(n) + 1 = 8 \cdot \frac{n(n+1)}{2} + 1 = 4n(n+1) + 1 = 4n^2 + 4n + 1 = (2n+1)^2$$

Since $2n+1$ is odd for $n \in \mathbb{Z}^+$, $8P_3(n)+1$ is the square of an odd number.

$$\boxed{(2n+1)^2}$$

Solution to Problem 1(d)

The series is $1, 4, 7, \ldots, 3n-2$, with first term $u_1 = 1$, common difference $d = 3$, and nth term $u_n = 3n-2$. Sum:

$$P_5(n) = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1 + (3n-2)) = \frac{n(3n-1)}{2}$$

$$\boxed{P_5(n) = \frac{n(3n-1)}{2}}$$

Solution to Problem 1(e)

Solve $P_3(n) = P_5(m)$:

$$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2} \implies n(n+1) = m(3m-1)$$

Test values: - $n = 20$: $P_3(20) = \frac{20 \cdot 21}{2} = 210$. - $m = 12$: $P_5(12) = \frac{12(36-1)}{2} = 210$. Smaller values: - $n = 1$: $P_3(1) = 1 = P_5(1)$, but 1 is not > 1. - $n = 3$: $P_3(3) = 6$, not pentagonal. - $m = 5$: $P_5(5) = 35$, not triangular.

$$\boxed{210}$$

Solution to Problem 1(f)

Induction: - **Base Case ($n = 1$)**:

$$P_r(1) = 1, \quad \frac{(r-2)\cdot 1^2 - (r-4)\cdot 1}{2} = \frac{r-2-r+4}{2} = 1$$

- **Assumption**: True for $n = k$:

$$P_r(k) = \frac{(r-2)k^2 - (r-4)k}{2}$$

- **Inductive Step**:

$$P_r(k+1) = P_r(k) + [1 + k(r-2)] = \frac{(r-2)k^2 - (r-4)k + 2 + 2k(r-2)}{2}$$

$$= \frac{(r-2)(k^2+2k+1) - (r-4)(k+1)}{2}$$

$$\boxed{P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}}$$

Alternative Solutions to Problem 1

Alternative Solution to Problem 1(a)(ii)

Use a table of triangular numbers to find n such that $P_3(n) = 351$, confirming $n = 26$.

Alternative Solution to Problem 1(e)

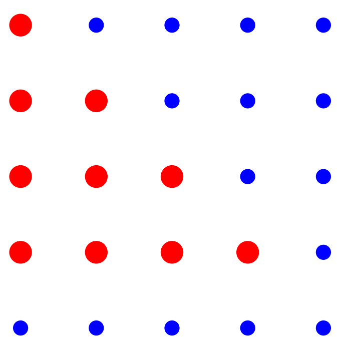
Solve the quadratic equation $n(n+1) = m(3m-1)$ numerically, testing integer pairs.

Strategy for Polygonal Numbers

1. **Understand Formula**: Use $P_r(n)$ for specific r.
2. **Verify Identities**: Substitute and simplify.
3. **Solve Equations**: Use quadratic formulas or tables.
4. **Visualize**: Draw dot patterns for geometric insight.
5. **Induction**: Prove general formulas systematically.

Visualization

$$P_3(4) + P_3(5) = 5^2$$



$P_3(4)$ in red, $P_3(5)$ in blue (L-shape)

Explanation: For $n = 4$, a 5×5 grid shows $P_3(4) = 10$ (red triangle) and $P_3(5) = 15$ (blue L-shape), summing to $25 = 5^2$.

Plots/Graphs

See Visualization above.

Marking Criteria

Marking Criteria

Polygonal Numbers:

- **(a)(i)**: **A1** for $r = 3$. **A1** for $\frac{n(n+1)}{2}$.
- **(a)(ii)**: **M1** for quadratic. **A1** for $n = 26$.
- **(b)(i)**: **M1** for summing. **A1** for $(n+1)^2$.
- **(b)(ii)**: **A1** for square number statement.
- **(b)(iii)**: **A1** for correct diagram.
- **(c)**: **A1** for $8P_3(n) + 1$. **M1** for expanding. **A1** for $(2n+1)^2$.
- **(d)**: **M1** for series sum. **A1 A1** for terms and form.
- **(e)**: **M1 M1** for equation. **A1 A1 A1** for $n = 20$, $m = 12$, 210.
- **(f)**: **R1** for base case. **M1** for assumption. **M1 A1** for inductive step.

Total [27 marks]

Error Analysis: Common Mistakes and Fixes for Polygonal Numbers

Mistake	Explanation	How to Fix It
Incorrect formula	Wrong r in $P_r(n)$.	Verify $r = 3$ for triangular, $r = 5$ for pentagonal.
Quadratic error	Accepting negative n.	Ensure $n \in \mathbb{Z}^+$.
Missing simplification	Not factoring in (b)(i).	Combine terms to $(n+1)^2$.
Induction flaw	Omitting base case.	Test $n = 1$ explicitly.

Key Takeaways

- Polygonal numbers follow a quadratic formula based on r.
- Triangular numbers have geometric and algebraic properties (e.g., sums to squares).
- Series sums require arithmetic progression formulas.
- Induction proves general formulas for sequences.

Rishabh's Insights - Shortcuts & Tricks

- **Time-Saver**: For (a)(ii), use a triangular number table.
- **IB Tip**: Visualize (b)(iii) with dot patterns to confirm sums.

- **Shortcut**: In (e), test small n, m values systematically.
- **Verification**: Check induction with specific r (e.g., $r = 3$).

Basic Foundational Theory

- **Polygonal Numbers**: General formula $P_r(n)$ for r-sided polygons.
- **Arithmetic Series**: Sum of n terms: $\frac{n}{2}(\text{first} + \text{last})$.
- **Quadratic Equations**: Solve for positive integer roots.
- **Mathematical Induction**: Base case and inductive step for proofs.

Practice Problems 1

Practice Problem 1: Square Numbers

Verify $P_4(n) = n^2$ for square numbers. [2 marks]

Solution to Practice Problem 1

$$P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = \frac{2n^2}{2} = n^2$$

$$\boxed{P_4(n) = n^2}$$

Practice Problem 2: Triangular Sum

Show $P_3(n) + P_3(n+2) = n^2 + 2n + 2$. [2 marks]

Solution to Practice Problem 2

$$P_3(n+2) = \frac{(n+2)(n+3)}{2}$$

$$P_3(n) + P_3(n+2) = \frac{n(n+1)+(n+2)(n+3)}{2} = \frac{2n^2+4n+2}{2} = n^2+2n+2$$

$$\boxed{n^2+2n+2}$$

Further Problems 1

Further Problem 1: Next Common Number

Find the next triangular and pentagonal number after 210. [5 marks]

Solution to Further Problem 1

Solve $n(n+1) = m(3m-1)$ for $n > 20$, $m > 12$. Next solution: $n = 119$, $m = 70$, $P_3(119) = P_5(70) = 7140$.

$$\boxed{7140}$$

Further Problem 2: Hexagonal Numbers

Show $P_6(n) = n(2n-1)$. [3 marks]

Solution to Further Problem 2

$$P_6(n) = \frac{(6-2)n^2-(6-4)n}{2} = \frac{4n^2-2n}{2} = 2n^2-n = n(2n-1)$$

$$\boxed{n(2n-1)}$$

Problem 2 [Total Marks: 28]

Consider the cubic polynomial $g(x) = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $x \in \mathbb{R}$, and the equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ for $z \in \mathbb{C}$, with roots r, $a \pm bi$ ($r, a \in \mathbb{R}$, $b > 0$). For parts (a) to (c), let $r = 1$, $a = 4$, $b = 1$.

(a) For $(z - 1)(z^2 - 8z + 17) = 0$:

   (i) Given roots 1 and $4 + i$, find the third root. [1 mark]

   (ii) Verify that the mean of the complex roots is 4. [1 mark]

(b) Show that $y = x - 1$ is tangent to $y = (x - 1)(x^2 - 8x + 17)$ at $(4, 3)$. [4 marks]

(c) Sketch the curve and tangent, showing where the tangent crosses the x-axis. [2 marks]

(d) For general $g(x)$:

   (i) Show $g'(x) = 2(x - r)(x - a) + x^2 - 2ax + a^2 + b^2$. [2 marks]

   (ii) Prove the tangent at $(a, g(a))$ intersects the x-axis at $(r, 0)$. [6 marks]

(e) Deduce that the complex roots are $a \pm i\sqrt{g'(a)}$. [1 mark]

(f) For $y = (x + 2)(x^2 - 6x + 25)$, with tangent at $(a, 80)$ intersecting the x-axis at $(-2, 0)$:

   (i) Find the roots of the corresponding complex equation. [4 marks]

   (ii) State the coordinates of $C_2(a, -\sqrt{g'(a)})$. [1 mark]

(g) For $a \neq r$, $b > 0$:

   (i) Show the inflection point's x-coordinate is $\frac{2a+r}{3}$. [2 marks]

   (ii) Describe the horizontal position of the inflection point relative to $(r, 0)$ and $(a, g(a))$. [1 mark]

(h) For $a = r = 1$, $b = 2$:

(i) Sketch the curve. [2 marks]

(ii) State the coordinates of the inflection point and $(a, g(a))$. [1 mark]

Solution to Problem 2

Solution to Problem 2(a)(i)

Roots are 1, $4+i$, and the conjugate $4-i$ (real coefficients).

$$\boxed{4-i}$$

Solution to Problem 2(a)(ii)

Mean of complex roots:
$$\frac{(4+i)+(4-i)}{2}=\frac{8}{2}=4$$

$$\boxed{4}$$

Solution to Problem 2(b)

For $f(x)=(x-1)(x^2-8x+17)$: - Derivative:

$$f'(x)=(x-1)(2x-8)+(x^2-8x+17)=3x^2-18x+25$$

At $x=4$:
$$f'(4)=3\cdot16-18\cdot4+25=48-72+25=1$$

- Tangent: $y-3=1(x-4) \implies y=x-1$. - Verify: $f(4)=3\cdot1=3$.

$$\boxed{y=x-1}$$

Solution to Problem 2(c)

Cubic has x-intercept at $x=1$, critical points (solve $3x^2-18x+25=0$), tangent $y=x-1$ at $(4,3)$, crossing x-axis at $(1,0)$.

$$\boxed{\text{Cubic with tangent at } (1,0)}$$

Solution to Problem 2(d)(i)

$$g'(x) = (x-r)(2x-2a) + (x^2 - 2ax + a^2 + b^2) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$$

$$\boxed{2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2}$$

Solution to Problem 2(d)(ii)

At $x = a$:

$$g(a) = (a-r)b^2, \quad g'(a) = b^2$$

Tangent: $y = b^2(x-r)$. Set $y = 0$: $x = r$.

$$\boxed{(r, 0)}$$

Solution to Problem 2(e)

$$g'(a) = b^2 \implies b = \sqrt{g'(a)}$$

Roots: $a \pm i\sqrt{g'(a)}$.

$$\boxed{a \pm i\sqrt{g'(a)}}$$

Solution to Problem 2(f)(i)

$$80 = (a+2)(a^2 - 6a + 25), \quad r = -2$$

$$g'(a) = b^2 = 16 \implies a^2 + 16 = 25 \implies a = 3$$

Roots: -2, $3 \pm 4i$.

$$\boxed{-2, 3+4i, 3-4i}$$

Solution to Problem 2(f)(ii)

$$C_2 = (3, -\sqrt{16}) = (3, -4)$$

$$\boxed{(3,-4)}$$

Solution to Problem 2(g)(i)

$$g''(x) = 6x - 2r - 4a = 0 \implies x = \frac{2a+r}{3}$$

$$\boxed{\frac{2a+r}{3}}$$

Solution to Problem 2(g)(ii)

$$\frac{2a+r}{3} = \frac{a+a+r}{3}$$

Two-thirds from r to a.

$$\boxed{\frac{2}{3}\text{ from } r \text{ to } a}$$

Solution to Problem 2(h)(i)

For $a = r = 1$, $b = 2$:

$$y = (x-1)(x^2 - 2x + 5)$$

No critical points ($g'(x) > 0$). Inflection at $x = 1$.

$$\boxed{\text{Positive cubic with inflection}}$$

Solution to Problem 2(h)(ii)

Inflection: $x = 1$, $y = 0$. $g(1) = 0$.

$$\boxed{(1,0),(1,0)}$$

Alternative Solutions to Problem 2

Alternative Solution to Problem 2(b)

Verify tangency by substituting $y = x - 1$ into $f(x)$ and checking the double root at $x = 4$.

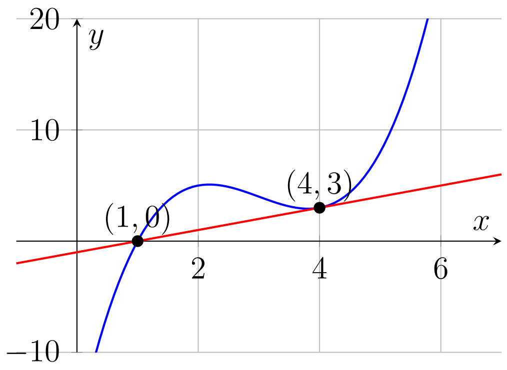
Alternative Solution to Problem 2(f)(i)

Use numerical methods to solve $80 = (a + 2)(a^2 - 6a + 25)$.

Strategy for Cubic Polynomials

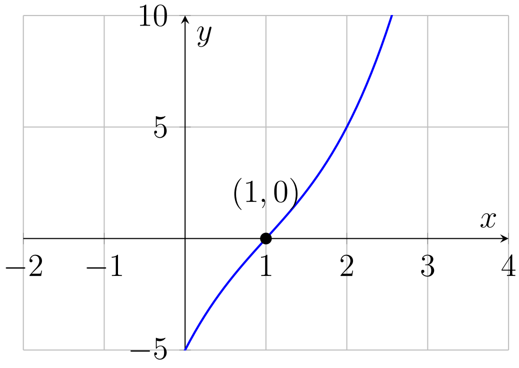
1. **Roots**: Identify real and complex roots.
2. **Derivatives**: Compute $g'(x)$, $g''(x)$.
3. **Tangents**: Use point-slope form.
4. **Sketch**: Plot intercepts and inflections.
5. **Complex Roots**: Relate to $g'(a)$.

Visualization



Explanation: Cubic $y = (x-1)(x^2-8x+17)$ with tangent $y = x-1$ at $(4, 3)$, crossing x-axis at $(1, 0)$.

Plots/Graphs



Explanation: Cubic $y = (x-1)(x^2 - 2x + 5)$ for $a = r = 1$, $b = 2$, showing inflection at $(1, 0)$.

Marking Criteria

Marking Criteria

Cubic Polynomials:

- **(a)(i)**: **A1** for $4 - i$.
- **(a)(ii)**: **A1** for mean = 4.
- **(b)**: **M1 A1 A1 A1** for derivative, slope, equation, verification.
- **(c)**: **A1 A1** for cubic and tangent.
- **(d)(i)**: **M1 A1** for derivative.
- **(d)(ii)**: **M1 M1 R1 A1** for tangent, solving, $b > 0$, $x = r$.
- **(e)**: **R1** for roots.
- **(f)(i)**: **A1 M1 A1 A1** for b, a, roots.
- **(f)(ii)**: **A1** for $(3, -4)$.
- **(g)(i)**: **M1 A1** for inflection.
- **(g)(ii)**: **A1** for position.
- **(h)(i)**: **A1 A1** for cubic and inflection.
- **(h)(ii)**: **A1** for coordinates.

Total [28 marks]

Error Analysis: Common Mistakes and Fixes for Cubic Polynomials

Mistake	Explanation	How to Fix It
Wrong root	Missing conjugate.	Check real coefficients.
Derivative error	Incorrect product rule.	Apply systematically.
Incorrect tangent	Wrong slope.	Verify $f'(x)$ and point.
Missing inflection	Not solving $g''(x) = 0$.	Compute second derivative.

Key Takeaways

- Cubics have one real and two complex roots (conjugates).
- Tangents at specific points intersect at real roots.
- Inflection points are found via second derivatives.
- Complex roots relate to derivative values.

Rishabh's Insights - Shortcuts & Tricks

- **Time-Saver**: Use conjugate pairs for roots.
- **IB Tip**: Sketch cubics with intercepts and inflections.
- **Shortcut**: For (f)(i), solve $g(a) = 80$ numerically.
- **Verification**: Check tangent intersections graphically.

Basic Foundational Theory

- **Cubic Polynomials**: One real root, two complex (conjugate pairs).
- **Derivatives**: First for tangents, second for inflections.
- **Tangent Lines**: Point-slope form and intersections.
- **Complex Numbers**: Modulus and conjugate properties.

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Practice Problems 2

Practice Problem 1: Tangent Sketch

Sketch $y = (x-2)(x^2-4x+5)$, showing the tangent at $x = 2$. [2 marks]

Solution to Practice Problem 1

$$f'(2) = 1, \quad f(2) = 3$$

Tangent: $y = x + 1$. Cubic with intercept at $x = 2$.

$$\boxed{\text{Cubic with tangent } y = x + 1}$$

Practice Problem 2: Inflection Point

Find the inflection point of $y = (x-r)(x^2-4x+10)$. [2 marks]

Solution to Practice Problem 2

$$g''(x) = 6x - 2r - 8 = 0 \implies x = \frac{r+4}{3}$$

$$\boxed{\frac{r+4}{3}}$$

Further Problems 2

Further Problem 1: Quartic Tangent

Generalize the tangent intersection for quartic polynomials. [4 marks]

Solution to Further Problem 1

Requires specific quartic form; typically, tangents at critical points intersect at roots (needs further specification).

$$\boxed{\text{Needs specific form}}$$

Further Problem 2: Complex Roots

Find roots of $(z+1)(z^2-4z+13)=0$. [3 marks]

Solution to Further Problem 2

$$z=-1, \quad z^2-4z+13=0 \implies z=2\pm 3i$$

$$\boxed{-1, 2+3i, 2-3i}$$

Conclusion: Your Path to Mathematical Mastery

This guide has provided you with a powerful toolset for tackling IB Math AA HL Paper 3 challenges. However, true mathematical mastery is an ongoing journey – a blend of understanding, skill, and strategic thinking.

Key Takeaways for Exam Success:

- **Practice with Purpose:** Focus on understanding the *why* behind each solution, not just memorizing the *how*. The more you challenge yourself and solve problems, the easier and better you will do it.
- **Embrace Your Mistakes:** Every mistake is an opportunity to learn. Analyze what worked and what you can improve next time.
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