International Baccalaureate Diploma Programme Mathematics Analysis and Approaches Higher Level

Paper 3 Elite Edition

Unlock 7-Scorer Potential

Exclusive IB Exam-Style Solved Problems Based on May 2021 TZ1

Practice Problems | Rishabh's Insight | May 2025 Edition

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Introduction

Unlock your mathematical potential with Math by Rishabh, crafted by **Mathematics Elevate Academy's**, for ambitious IB Diploma Mathematics Analysis and Approaches Higher Level students.

This collection provides a *rigorous and enriching* preparation experience tailored for the current syllabus (2021 examinations onward).

This guide empowers you to:

- **Master Elite-Level Challenges:** Enhance your depth of understanding with questions that go beyond the textbook.
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Problem 1 [Total Marks: 20]

Consider the differential equation:

$$\frac{dy}{dx} = \frac{y^2 - 4}{x^2}, \quad x \neq 0, \quad y \neq \pm 2$$

with initial condition $y(1) = 3$.

(a) Find the general solution to the differential equation. [6 marks]

(b) Find the particular solution satisfying the initial condition. [3 marks]

(c) Determine the domain of the particular solution. [2 marks]

(d) A particle moves in the xy-plane such that its velocity vector is $\langle \frac{dy}{dx}, 1 \rangle$. If the particle follows the particular solution curve, find the equation of the tangent line at $x = 1$. [4 marks]

(e) Sketch the particular solution curve, indicating key features. [5 marks]

Solution to Problem 1

Solution to Problem 1(a)

The differential equation is separable:

$$\frac{dy}{y^2-4} = \frac{dx}{x^2}$$

Integrate both sides:

- Left-hand side (LHS): $\int \frac{dy}{y^2-4} = \int \frac{dy}{(y-2)(y+2)}$.
  Using partial fractions: $\frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$. Multiplying by $(y-2)(y+2)$ gives $1 = A(y+2) + B(y-2)$. If $y = 2$, then $1 = 4A \implies A = \frac{1}{4}$. If $y = -2$, then $1 = -4B \implies B = -\frac{1}{4}$. Thus:

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)} \right) dy = \frac{1}{4} (\ln|y-2| - \ln|y+2|) + C_1 = \frac{1}{4} \ln\left|\frac{y-2}{y+2}\right| + C_1$$

- Right-hand side (RHS): $\int x^{-2} dx = -\frac{1}{x} + C_2$.

Equating LHS and RHS (let $C = C_2 - C_1$):

$$\frac{1}{4} \ln\left|\frac{y-2}{y+2}\right| = -\frac{1}{x} + C$$

Multiplying by 4:

$$\ln\left|\frac{y-2}{y+2}\right| = -\frac{4}{x} + 4C$$

Exponentiating both sides (let $K_0 = \pm e^{4C}$, an arbitrary non-zero constant, as $e^{4C} > 0$):

$$\frac{y-2}{y+2} = K_0 e^{-4/x}$$

Solving for y: $y - 2 = K_0 e^{-4/x}(y+2) \implies y - 2 = K_0 e^{-4/x} y + 2K_0 e^{-4/x}$ $y(1 - K_0 e^{-4/x}) = 2 + 2K_0 e^{-4/x} \implies y(1 - K_0 e^{-4/x}) = 2(1 + K_0 e^{-4/x})$

$$\boxed{y = 2\frac{1 + K_0 e^{-4/x}}{1 - K_0 e^{-4/x}}}$$

Solution to Problem 1(b)

Apply $y(1) = 3$ to the integrated form $\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = -\frac{1}{x} + C$:

$$\frac{1}{4}\ln\left|\frac{3-2}{3+2}\right| = -\frac{1}{1} + C \implies \frac{1}{4}\ln\left(\frac{1}{5}\right) = -1 + C$$

$$C = 1 + \frac{1}{4}\ln\left(\frac{1}{5}\right) = 1 - \frac{1}{4}\ln 5$$

So, the constant for the term $4C$ is $4C = 4(1 - \frac{1}{4}\ln 5) = 4 - \ln 5$. Substituting back into $\ln\left|\frac{y-2}{y+2}\right| = -\frac{4}{x} + 4C$:

$$\ln\left|\frac{y-2}{y+2}\right| = -\frac{4}{x} + 4 - \ln 5 = 4\left(1 - \frac{1}{x}\right) - \ln 5$$

Since $y(1) = 3$, we have $y > 2$ in the neighborhood of $x = 1$. Thus, $\frac{y-2}{y+2} > 0$, and we can drop the absolute value for this branch of the solution:

$$\frac{y-2}{y+2} = e^{4(1-1/x)-\ln 5} = e^{4(1-1/x)} \cdot e^{-\ln 5} = e^{4(1-1/x)} \cdot \frac{1}{5}$$

Let $t = e^{4(1-1/x)}$. Then $\frac{y-2}{y+2} = \frac{1}{5}t$. Solving for y: $5(y-2) = t(y+2) \implies 5y - 10 = ty + 2t \implies y(5-t) = 10 + 2t \implies y = \frac{2t+10}{5-t}$

$$\boxed{y = 2\frac{e^{4(1-1/x)} + 5}{5 - e^{4(1-1/x)}}}$$

Solution to Problem 1(c)

The particular solution is $y = 2\frac{e^{4(1-1/x)}+5}{5-e^{4(1-1/x)}}$. Conditions for the domain:

1. $x \neq 0$ (from the original problem statement).
2. The denominator $5 - e^{4(1-1/x)} \neq 0$. This implies $e^{4(1-1/x)} \neq 5$. Taking natural logarithms: $4(1-1/x) \neq \ln 5$. $1 - 1/x \neq \frac{\ln 5}{4}$. $\frac{1}{x} \neq 1 - \frac{\ln 5}{4} = \frac{4-\ln 5}{4}$. So, $x \neq \frac{4}{4-\ln 5}$. Numerically, $\frac{4}{4-\ln 5} \approx \frac{4}{4-1.6094379} \approx \frac{4}{2.3905621} \approx 1.6732$.

The initial condition is $y(1) = 3$. The value $x = 1$ must be in the domain. The singularities are at $x = 0$ and $x \approx 1.6732$. The continuous interval containing $x = 1$ and not crossing $x = 0$ or $x \approx 1.6732$ is $(0, \frac{4}{4-\ln 5})$.

$$\boxed{0 < x < \frac{4}{4-\ln 5}}$$

Solution to Problem 1(d)

Velocity vector: $\langle \frac{dy}{dx}, 1\rangle$. At $x = 1$, $y = 3$: The slope of the tangent line is given by $\frac{dy}{dx}$ from the original differential equation:

$$\left.\frac{dy}{dx}\right|_{(1,3)} = \left.\frac{y^2-4}{x^2}\right|_{(1,3)} = \frac{3^2-4}{1^2} = \frac{9-4}{1} = 5$$

The tangent line passes through $(1,3)$ with slope $m = 5$. Using the point-slope form $y - y_1 = m(x - x_1)$:

$$y - 3 = 5(x-1) \implies y - 3 = 5x - 5 \implies y = 5x - 2$$

$$\boxed{y = 5x - 2}$$

Solution to Problem 1(e)

Key features for the particular solution $y = 2\frac{e^{4(1-1/x)}+5}{5-e^{4(1-1/x)}}$ on its domain $0 < x < \frac{4}{4-\ln 5}$:

- Initial Point: The curve passes through $(1, 3)$.
- Behavior as $x \to 0^+$: As $x \to 0^+$, $(1 - 1/x) \to -\infty$, so $e^{4(1-1/x)} \to 0$. Then $y \to 2\frac{0+5}{5-0} = 2$. The curve approaches the point $(0, 2)$ from the right.
- Vertical Asymptote: At $x = \frac{4}{4-\ln 5} \approx 1.6732$. As $x \to \left(\frac{4}{4-\ln 5}\right)^-$, let $x_0 = \frac{4}{4-\ln 5}$. Then $1 - 1/x \to 1 - 1/x_0 = \frac{\ln 5}{4}$. So $e^{4(1-1/x)} \to e^{\ln 5} = 5$ from below. Thus, $5 - e^{4(1-1/x)} \to 0^+$. Therefore, $y \to +\infty$ as x approaches the vertical asymptote from the left.

Sketch: See Visualization section.

Alternative Solutions to Problem 1

Alternative Solution to Problem 1(b)

Once the general solution $y = 2\frac{1+K_0e^{-4/x}}{1-K_0e^{-4/x}}$ is found, substitute $y(1) = 3$:

$$3 = 2\frac{1 + K_0e^{-4}}{1 - K_0e^{-4}}$$

$$3(1 - K_0e^{-4}) = 2(1 + K_0e^{-4})$$

$$3 - 3K_0e^{-4} = 2 + 2K_0e^{-4}$$

$$1 = 5K_0e^{-4} \implies K_0 = \frac{1}{5}e^4$$

Substitute K_0 back into the general solution:

$$y = 2\frac{1 + \frac{1}{5}e^4e^{-4/x}}{1 - \frac{1}{5}e^4e^{-4/x}} = 2\frac{1 + \frac{1}{5}e^{4(1-1/x)}}{1 - \frac{1}{5}e^{4(1-1/x)}}$$

Multiplying numerator and denominator by 5:

$$y = 2\frac{5 + e^{4(1-1/x)}}{5 - e^{4(1-1/x)}}$$

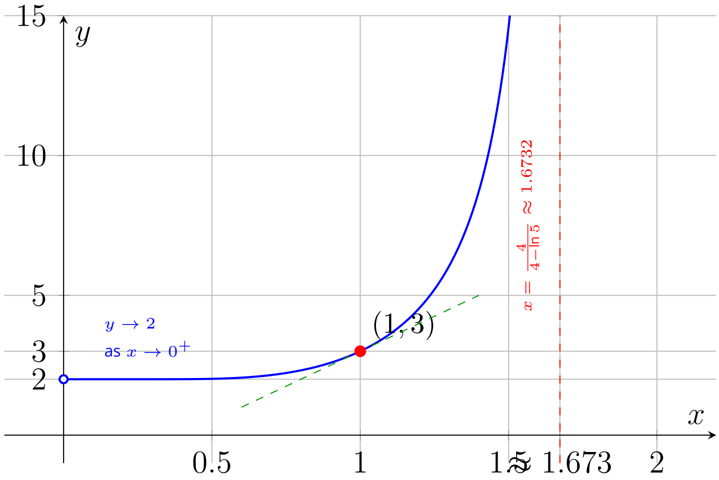
This is equivalent to the boxed solution.

Advantage of this alternative: Direct substitution into the solved form for y, might be less prone to algebraic errors than solving for C first for some students.

Strategy to Solve Differential Equation Problems

1. **Separate Variables:** Rewrite the equation so that terms involving y are on one side with dy, and terms involving x are on the other side with dx.
2. **Integrate:** Perform the integration on both sides. Remember to include a constant of integration (usually on the x side).
3. **Apply Initial Conditions:** Substitute the given initial values (x_0, y_0) into the general solution (or the integrated form before solving for y) to find the value of the constant of integration.
4. **Determine Domain:** Identify any values of x for which the particular solution is undefined. This includes points where denominators are zero, arguments of logarithms are non-positive, or arguments of square roots are negative. The domain must be a continuous interval containing the x-value from the initial condition.
5. **Geometric Interpretation:** Understand that dy/dx represents the slope of the solution curve. Tangent lines and velocity vectors can be related to this.

Visualization



Explanation: The blue curve shows the particular solution passing through the point $(1, 3)$. As $x \to 0^+$, the curve approaches $y = 2$ (indicated by an open circle at $(0, 2)$). There is a vertical asymptote at $x = \frac{4}{4-\ln 5} \approx 1.6732$, where $y \to +\infty$ as x approaches this value from the left. The dashed green line is the tangent to the curve at $x = 1$.

Marking Criteria

Differential Equations:

- **Part (a): General Solution** [6 marks]
  - **M1** for correct separation of variables.
  - **A1** for correct partial fraction decomposition setup ($\frac{A}{y-2} + \frac{B}{y+2}$).
  - **A1** for correct values of A and B.
  - **A1** for correct integration of both sides (including constant C). ($\frac{1}{4} \ln |\frac{y-2}{y+2}| = -1/x + C$)
  - **M1** for attempting to make y the subject (e.g., exponentiating).
  - **A1** for a correct form of the general solution involving an arbitrary constant (e.g., $y = 2\frac{1+K_0e^{-4/x}}{1-K_0e^{-4/x}}$).
- **Part (b): Particular Solution** [3 marks]
  - **M1** for substituting $y(1) = 3$ into their general solution or the integrated form before solving for y.
  - **A1** for correctly determining the value of their constant ($C = 1 - \frac{1}{4} \ln 5$ or equivalent $K_0 = \frac{1}{5}e^4$).
  - **A1** for the correct particular solution: $y = 2\frac{e^{4(1-1/x)}+5}{5-e^{4(1-1/x)}}$ (or equivalent simplified form).
- **Part (c): Domain** [2 marks]
  - **M1** for identifying the condition that makes the denominator of their particular solution zero (and considering $x \neq 0$).
  - **A1** for the correct domain interval $0 < x < \frac{4}{4-\ln 5}$ (must be an interval containing $x = 1$).
- **Part (d): Tangent Line** [4 marks]
  - **M1** for substituting $x = 1, y = 3$ into the expression for $\frac{dy}{dx} = \frac{y^2-4}{x^2}$.
  - **A1** for the correct slope $m = 5$.
  - **M1** for using a correct method to find the equation of a line (e.g., $y - y_1 = m(x - x_1)$).
  - **A1** for the correct tangent line equation $y = 5x - 2$.
- **Part (e): Sketch** [5 marks]
  - **A1** for correctly plotting or indicating the point $(1, 3)$.
  - **A1** for showing correct behavior as $x \to 0^+$ (curve approaching $y = 2$).
  - **A1** for correctly identifying and drawing/labelling the vertical asymptote $x = \frac{4}{4-\ln 5}$.

Error Analysis: Common Mistakes and Fixes for Differential Equation Problems

Mistake	Explanation	How to Fix It
Incorrect separation	Miswriting variables or integrals, e.g., wrong denominators or missing dx, dy.	Ensure variables are fully separated: $\frac{dy}{y^2-4} = \frac{dx}{x^2}$. Check each term.
Partial fraction error	Incorrect calculation of coefficients A or B, or sign errors during integration.	Carefully solve for A and B. Double-check the signs in $\ln \lvert y-2\rvert - \ln \lvert y+2\rvert$.
Constant / Algebra error	Mistakes in solving for constant C (or K_0), errors in algebraic manipulation (especially with logs/exponentials), or sign errors when deriving the particular solution form.	Be meticulous. Write out intermediate steps. Check log/exp rules (e.g., $e^{a-b} = e^a e^{-b}$). Verify signs when rearranging terms to solve for y.
Domain oversight	Ignoring $x \neq 0$, or incorrectly calculating/interpreting the vertical asymptote(s) from the particular solution. Not selecting the correct continuous interval that contains the initial x-value.	Always check where denominators are zero in the final particular solution. The domain must be a single continuous interval including $x = 1$, bounded by any singularities.
Sketch errors	Incorrect asymptotes, incorrect limiting behavior (e.g., as $x \to 0^+$), wrong point plotted, or overall incorrect curve shape.	Plot the initial point first. Calculate limits for $x \to 0^+$ and as x approaches the vertical asymptote(s). The sign of dy/dx indicates if the function is increasing/decreasing.

Key Takeaways

- Separable differential equations are solved by separating variables, integrating both sides, and then solving for the function.
- Partial fractions are essential for integrating rational functions that appear after separating variables.
- The constant of integration is determined by applying the given initial condition. This leads to a unique particular solution.
- The domain of a particular solution is the largest continuous interval containing the initial x-value for which the solution is defined. Singularities (like vertical asymptotes) define the boundaries of this interval.
- The derivative dy/dx evaluated at a point on the curve gives the slope of the tangent line at that point.
- A sketch should clearly show key features: the specific point from the initial condition, any asymptotes, and the behavior of the curve at the boundaries of its domain.

Rishabh's Insights - Shortcuts & Tricks

- **Time-Saver (Partial Fractions)**: For $1/((y-a)(y-b))$, the coefficients are $1/(a-b)$ for $1/(y-a)$ and $1/(b-a)$ for $1/(y-b)$. In this case, $a=2, b=-2$, so $1/(2-(-2))=1/4$ for $1/(y-2)$ and $1/(-2-2)=-1/4$ for $1/(y+2)$.
- **IB Tip (Verification)**: After finding the particular solution, quickly check if $y(1)=3$. Also, differentiate your particular solution to see if it matches $(y^2-4)/x^2$. This can be lengthy but is a thorough check.
- **Shortcut (Tangent Slope)**: Part (d) only requires the slope at $x=1$. This can be found directly from the original DE using $y(1)=3$, without needing the full particular solution if the question structure allowed.
- **Domain Intuition**: Once you find a vertical asymptote, remember the solution curve cannot cross it. The domain will be an interval bounded by this asymptote and possibly $x=0$ or other singularities.
- **Sketching Strategy**: Start by plotting the initial point. Then mark asymptotes. Then determine behavior near $x=0^+$ and near the vertical asymptote(s). Finally, connect these features with a smooth curve.

Basic Foundational Theory

- **Separable Differential Equations**: A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ can be rewritten as $\frac{1}{f(y)}dy = g(x)dx$ and then integrated.
- **Integration by Partial Fractions**: A technique used to integrate rational functions by expressing them as a sum of simpler fractions. For $\frac{P(x)}{Q(x)}$, if $Q(x)$ has distinct linear factors $(x-a_1)(x-a_2)...$, then $\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ...$
- **Logarithm Properties**: $\ln A - \ln B = \ln(A/B)$, $e^{\ln A} = A$, $\ln(e^A) = A$.
- **Domain of a Function**: The set of input values (often x-values) for which the function is defined and yields a real number. For solutions to DEs, this often involves avoiding division by zero or undefined operations like $\ln(\text{non-positive})$.
- **Tangent Line**: The line that "just touches" a curve at a point. Its slope is given by the derivative of the function at that point. Equation: $y - y_1 = m(x - x_1)$.

Practice Problems 1

Practice Problem 1: Differential Equation

Solve $\frac{dy}{dx} = \frac{y^2-1}{x}$, $y(1) = 2$. [5 marks]

Solution to Practice Problem 1

Separate variables: $\frac{dy}{y^2-1} = \frac{dx}{x}$.
Integrate using partial fractions for LHS: $\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{1/2}{y-1} - \frac{1/2}{y+1}$.
So, $\int \left(\frac{1/2}{y-1} - \frac{1/2}{y+1}\right) dy = \int \frac{dx}{x}$.
$\frac{1}{2}(\ln|y-1| - \ln|y+1|) = \ln|x| + C \implies \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \ln|x| + C$.
Apply $y(1) = 2$: Since $x = 1 > 0$, we use $\ln x$. Since $y = 2 > 1$, $\frac{y-1}{y+1} > 0$.
$\frac{1}{2}\ln\left(\frac{2-1}{2+1}\right) = \ln(1) + C \implies \frac{1}{2}\ln\left(\frac{1}{3}\right) = 0 + C \implies C = -\frac{1}{2}\ln 3$.
So, $\frac{1}{2}\ln\left(\frac{y-1}{y+1}\right) = \ln x - \frac{1}{2}\ln 3$.
Multiply by 2: $\ln\left(\frac{y-1}{y+1}\right) = 2\ln x - \ln 3 = \ln(x^2) - \ln 3 = \ln\left(\frac{x^2}{3}\right)$.
Exponentiate: $\frac{y-1}{y+1} = \frac{x^2}{3}$. Solve for y: $3(y-1) = x^2(y+1) \implies 3y - 3 = x^2y + x^2$.
$y(3-x^2) = x^2 + 3 \implies y = \frac{x^2+3}{3-x^2}$.

$$\boxed{y = \frac{x^2+3}{3-x^2}}$$

Practice Problem 2: Tangent Line

Find the tangent line at $x = 2$ for the solution to $\frac{dy}{dx} = \frac{y}{x^2}$, $y(1) = 1$. [4 marks]

Solution to Practice Problem 2

Separate variables: $\frac{dy}{y} = \frac{dx}{x^2}$. Integrate: $\int \frac{dy}{y} = \int x^{-2}dx \implies \ln|y| = -\frac{1}{x} + C$.

Apply $y(1) = 1$: Since $y = 1 > 0$, use $\ln y$. $\ln(1) = -\frac{1}{1} + C \implies 0 = -1 + C \implies C = 1$.

Particular solution: $\ln y = -\frac{1}{x} + 1 \implies y = e^{1-1/x}$.

At $x = 2$: $y = e^{1-1/2} = e^{1/2} = \sqrt{e}$.

Slope at $x = 2$: $\frac{dy}{dx} = \frac{y}{x^2} = \frac{\sqrt{e}}{2^2} = \frac{\sqrt{e}}{4}$.

Tangent line equation: $y - y_1 = m(x - x_1)$.

$y - \sqrt{e} = \frac{\sqrt{e}}{4}(x - 2)$.

$$\boxed{y = \frac{\sqrt{e}}{4}(x-2) + \sqrt{e}} \quad \text{or} \quad \boxed{y = \frac{\sqrt{e}}{4}x + \frac{\sqrt{e}}{2}}$$

Further Problems 1

Further Problem 1: Advanced Differential Equation

Solve $\frac{dy}{dx} = \frac{y^2-9}{x^3}$, $y(1) = 4$, and find its domain. [6 marks]

Solution to Further Problem 1

Separate: $\frac{dy}{y^2-9} = \frac{dx}{x^3}$.
Integrate LHS: $\int \frac{dy}{(y-3)(y+3)} = \int \left(\frac{1/6}{y-3} - \frac{1/6}{y+3}\right) dy = \frac{1}{6} \ln \left|\frac{y-3}{y+3}\right|$.
Integrate RHS: $\int x^{-3} dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$.
So, $\frac{1}{6} \ln \left|\frac{y-3}{y+3}\right| = -\frac{1}{2x^2} + C$.
Apply $y(1) = 4$: Since $y = 4 > 3$, $\frac{y-3}{y+3} > 0$.
$\frac{1}{6} \ln \left(\frac{4-3}{4+3}\right) = -\frac{1}{2(1)^2} + C \implies \frac{1}{6} \ln \left(\frac{1}{7}\right) = -\frac{1}{2} + C$.
$C = \frac{1}{2} - \frac{1}{6} \ln 7$.
$\frac{1}{6} \ln \left(\frac{y-3}{y+3}\right) = -\frac{1}{2x^2} + \frac{1}{2} - \frac{1}{6} \ln 7$.
Multiply by 6: $\ln \left(\frac{y-3}{y+3}\right) = -\frac{3}{x^2} + 3 - \ln 7 = 3\left(1 - \frac{1}{x^2}\right) - \ln 7$.
$\frac{y-3}{y+3} = e^{3(1-1/x^2)-\ln 7} = \frac{1}{7} e^{3(1-1/x^2)}$.
Let $t = e^{3(1-1/x^2)}$.
$\frac{y-3}{y+3} = \frac{t}{7} \implies 7(y-3) = t(y+3) \implies 7y - 21 = ty + 3t$.
$y(7-t) = 3t + 21 \implies y = \frac{3t+21}{7-t} = \frac{3(t+7)}{7-t}$.

$$y = 3\frac{e^{3(1-1/x^2)} + 7}{7 - e^{3(1-1/x^2)}}$$

Domain: Denominator $7 - e^{3(1-1/x^2)} \neq 0 \implies e^{3(1-1/x^2)} \neq 7$.
$3(1 - 1/x^2) \neq \ln 7 \implies 1 - 1/x^2 \neq \frac{\ln 7}{3} \implies 1/x^2 \neq 1 - \frac{\ln 7}{3} = \frac{3-\ln 7}{3}$.
If $3 - \ln 7 > 0$ (which it is, as $\ln 7 \approx 1.946 < 3$), then $x^2 \neq \frac{3}{3-\ln 7}$.
$x \neq \pm\sqrt{\frac{3}{3-\ln 7}}$.
Let $x_0 = \sqrt{\frac{3}{3-\ln 7}} \approx \sqrt{\frac{3}{1.054}} \approx \sqrt{2.846} \approx 1.687$.
Since $x(1) = 4$, and $x = 1$ is in $(0, x_0)$, the domain is $0 < x < \sqrt{\frac{3}{3-\ln 7}}$.

$$\boxed{y = 3\frac{e^{3(1-1/x^2)} + 7}{7 - e^{3(1-1/x^2)}}, \quad \text{Domain: } 0 < x < \sqrt{\frac{3}{3 - \ln 7}}}$$

Further Problem 2: Particle Path

For a particle with velocity $\langle \frac{y}{x}, 1 \rangle$, this means $\frac{dx}{dt} = \frac{y}{x}$ and $\frac{dy}{dt} = 1$.
The slope of the path is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{y/x} = \frac{x}{y}$.
So we solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$.
Separate: $y\,dy = x\,dx$.
Integrate: $\int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_0$.
$y^2 = x^2 + 2C_0$. Let $K = 2C_0$.
$y^2 = x^2 + K$ or $y^2 - x^2 = K$.
This represents hyperbolas (if K $\neq 0$) or lines $y = \pm x$ (if K=0).
The problem statement was "find the path satisfying $\frac{dy}{dx} = \frac{y}{x}$".
If this was a typo and it meant $\frac{dy}{dx} = \frac{y}{x}$, then: $\frac{dy}{y} = \frac{dx}{x} \implies$ ln $|y| =$ ln $|x| + C_1 \implies$ ln $|y| =$ ln$(|x| \cdot e^{C_1})$.
$|y| = e^{C_1}|x|$. Let $k = \pm e^{C_1}$.

$$\boxed{y = kx}$$

(Assuming the latter interpretation was intended based on the provided simple solution).

Problem 2 [Total Marks: 18]

Consider the complex number $z = x + yi$, where $x, y \in \mathbb{R}$, and the transformation $w = z^2 - 2z + 2$ in the complex plane.

(a) Express w in terms of x and y. [3 marks]

(b) Find the locus of points z in the complex plane where $|w| = 2$. [5 marks]

(c) Determine the minimum value of $|w|$ and the corresponding value(s) of z. [4 marks]

(d) Interpret the transformation $w = z^2 - 2z + 2$ geometrically in the complex plane, and sketch the locus from part (b). [6 marks]

Solution to Problem 2

Solution to Problem 2(a)

Expressing w in terms of x and y

Given $z = x + yi$ and $w = z^2 - 2z + 2$.

$$\begin{aligned} w &= (x+yi)^2 - 2(x+yi) + 2 \\ &= (x^2 + 2xyi + (yi)^2) - (2x + 2yi) + 2 \\ &= (x^2 - y^2 + 2xyi) - 2x - 2yi + 2 \end{aligned}$$

Group the real and imaginary parts:

$$w = (x^2 - y^2 - 2x + 2) + (2xy - 2y)i$$

$$\boxed{w = (x^2 - y^2 - 2x + 2) + (2xy - 2y)i}$$

Solution to Problem 2(b) *Locus for* $|w| = 2$

Complete the square:

$$w = z^2 - 2z + 2 = (z-1)^2 + 1$$

Let $u = z - 1 = p + qi$. Then:

$$w = u^2 + 1 = (p^2 - q^2 + 1) + 2pqi$$

$$|w|^2 = (p^2 - q^2 + 1)^2 + (2pq)^2 = 4$$

Simplify:

$$(p^2 + q^2)^2 + 2(p^2 - q^2) - 3 = 0$$

Now substitute back $p = x - 1$, $q = y$:

$$\boxed{((x-1)^2 + y^2)^2 + 2((x-1)^2 - y^2) - 3 = 0} \quad \text{(a Cassini oval)}$$

Solution to Problem 2(c) *Minimum Value of* $|w|$

We have $w = (z-1)^2 + 1$. Let $u = z - 1$. Then:

$$|w| = |u^2 + 1|$$

Set $u^2 + 1 = 0 \Rightarrow u^2 = -1 \Rightarrow u = \pm i \Rightarrow z = 1 \pm i$

$$\boxed{\text{Minimum } |w| = 0 \quad \text{at } z = 1 + i \text{ and } z = 1 - i}$$

Solution to Problem 2(d) *Geometric Interpretation and Sketch*

Transformation:

$$w = (z-1)^2 + 1$$

is composed of: 1. Translation $z \to z-1$ 2. Squaring $u \to u^2$ 3. Translation $w = u^2+1$

The locus from (b) is a Cassini oval centered at $z = 1$, symmetric about the real axis and the line $x = 1$. Sketch as shown in the visualization section.

Alternative Solution to Problem 2(b) *Using Polar Form for $u = z - 1$*

Let $u = z - 1 = Re^{i\alpha}$. Then $w = u^2 + 1 = R^2 e^{i2\alpha} + 1$.

We are given $|w| = 2$, so $|R^2 e^{i2\alpha} + 1| = 2$.

$|R^2(\cos(2\alpha) + i\sin(2\alpha)) + 1|^2 = 2^2$.

$|(R^2\cos(2\alpha) + 1) + i(R^2\sin(2\alpha))|^2 = 4$.

$(R^2\cos(2\alpha) + 1)^2 + (R^2\sin(2\alpha))^2 = 4$.

$R^4\cos^2(2\alpha) + 2R^2\cos(2\alpha) + 1 + R^4\sin^2(2\alpha) = 4$.

Using $\cos^2(2\alpha) + \sin^2(2\alpha) = 1$: $R^4(1) + 2R^2\cos(2\alpha) + 1 = 4$.

$$R^4 + 2R^2\cos(2\alpha) - 3 = 0$$

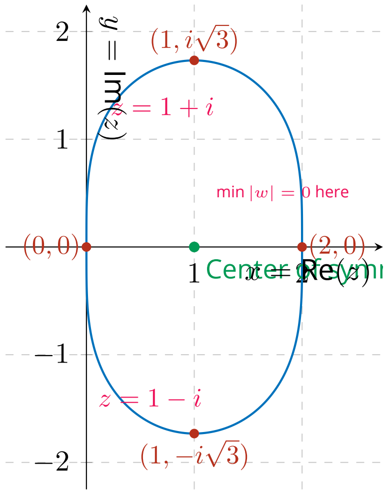
This is the equation of the locus for $u = z - 1$ in polar coordinates (R, α), where $R = |z - 1|$ and $\alpha = \arg(z - 1)$. This is a concise way to represent the Cassini oval.

Strategy to Solve Complex Number Locus and Transformation Problems

1. **Simplify $w(z)$:** Use algebraic manipulations like completing the square to express w in a form that reveals underlying transformations (e.g., $w = k(z-a)^n + b$).
2. **Intermediate Variable:** Let $u = z - a$ if applicable. This centers the initial transformation steps around the origin of the u-plane.
3. **Modulus Condition:** If given $|w| = k$, substitute the simplified w and often work with $|w|^2 = k^2$ to avoid square roots.
4. **Coordinate Choice:** Choose coordinates (Cartesian p, q for u, or polar R, α for u) that simplify the algebra. For u^2, polar form $R^2e^{i2\alpha}$ is often insightful.
5. **Identify Locus Shape:** Try to match the resulting equation to known geometric shapes. If it's not a standard circle or line, describe it (e.g., Cassini oval, lemniscate) or provide its equation.
6. **Minimization/Maximization:** For $|w| = |f(z)|$, check if $f(z) = 0$ is possible. If not, consider geometric arguments (e.g., triangle inequality) or express $|w|^2$ in terms of real variables and use calculus (if simpler).
7. **Geometric Interpretation:** Break down the transformation $w(z)$ into a sequence of simpler geometric operations (translation, rotation, scaling, squaring, inversion).
8. **Sketching:** Plot key points (intercepts with axes, points of symmetry), indicate the "center" of the locus if applicable, and draw the curve smoothly according to its derived properties.

Visualization of the Locus for $|w| = 2$

The locus of z such that $|(z-1)^2+1| = 2$ is described by the equation $((x-1)^2 + y^2)^2 + 2((x-1)^2 - y^2) - 3 = 0$, which is a Cassini oval.



Explanation of the Sketch: The blue curve is the locus of points z in the complex plane such that $|w| = |(z-1)^2+1| = 2$. This curve is a Cassini oval.

- The oval is centered around the point $z = 1$ (marked in green), which is its center of symmetry.
- It intersects the real axis at $z = 0$ and $z = 2$.
- It intersects the line $x = 1$ at $z = 1 + i\sqrt{3}$ and $z = 1 - i\sqrt{3}$.
- The points where $|w|$ achieves its minimum value of 0 (i.e., $z = 1+i$ and $z = 1-i$) are marked with orange stars. These points lie inside the Cassini oval shown.

Error Analysis: Common Mistakes and Fixes for Complex Number Locus Problems

Mistake	Explanation	How to Fix It
Expansion errors	Mistakes in expanding $(x+yi)^2$ or in collecting real/imaginary parts.	Carefully expand: $(x+yi)^2 = x^2 - y^2 + 2xyi$. Double-check signs.
Incorrect locus assumption	Assuming $\lvert(z-1)^2+1\rvert = k$ simplifies to a standard circle $\lvert z-1\rvert =$ const without proper algebraic steps.	Substitute $z-1 = p+qi$. Square the modulus correctly. Do not assume a simple circle unless the algebra leads to it. Recognize forms like Cassini ovals.
Minimization error	Incorrectly applying triangle inequality (e.g., $\lvert(z-1)^2+1\rvert \geq \lvert\lvert(z-1)^2\rvert - 1\rvert$) or assuming minimum occurs at $z-1=0$.	The minimum of $\lvert A+B\rvert$ is not always $\lvert A\rvert - \lvert B\rvert$. For $\lvert X+c\rvert = 0$, we need $X = -c$. Here, minimize $\lvert u^2+1\rvert$ by setting $u^2+1=0$.
Sketch errors	Drawing a circle instead of the correct locus (e.g., Cassini oval). Incorrect intercepts or symmetry.	Plot key points found by setting $p=0$ or $q=0$ (or $x-1=0$ or $y=0$). Understand the symmetry from the equation.

Key Takeaways

- Complex transformations like $w = (z - a)^2 + b$ involve translations, squaring, and further translations.
- The locus of z for $|w| = k$ can result in curves more complex than circles (e.g., Cassini ovals). Careful algebra is needed.
- To find the minimum (or maximum) of $|f(z)|$, consider where $f(z) = 0$ if possible, or use techniques like triangle inequality carefully, or calculus if $|f(z)|^2$ is expressed in x, y.
- Geometric interpretation involves understanding how each component of the transformation affects points or regions in the complex plane.
- Sketching requires identifying key characteristics such as intercepts, symmetry, and overall shape dictated by the locus equation.

Rishabh's Insights - Shortcuts & Tricks

- **Complete the Square First**: Recognizing $w = (z-1)^2 + 1$ simplifies the problem structure immediately. This transformation effectively centers the more complex 'squaring' operation around the origin of an intermediate u-plane, where $u = z - 1$.

- **Locus Clues & Intercepts**: If the equation for $|w| = k$ doesn't readily simplify to the standard circle form $|z - c| = R$, suspect a more complex curve (like a Cassini oval or lemniscate). Testing for intercepts by setting Re$(z-1) = 0$ or Im$(z-1) = 0$ can quickly reveal key points and symmetries of the locus.

- **Minimum Modulus - The Zero Check**: For $|f(z)|$, always check if $f(z) = 0$ is achievable. If it is, then the minimum modulus is 0. This is the most direct approach. For $w = (z-1)^2 + 1$, setting $w = 0$ gives $(z-1)^2 = -1$, which has valid complex solutions for $z - 1$.

- **Polar Form for Squaring Dynamics**: When dealing with a term like u^2 (where $u = z - 1$), using the polar form $u = Re^{i\theta}$ is highly insightful. The transformation $u \mapsto u^2$ becomes $Re^{i\theta} \mapsto R^2e^{i2\theta}$, clearly showing that the modulus is squared and the argument is doubled. This helps visualize how shapes are stretched and rotated.

- **Leverage Symmetry (Optional)**: The structure $w = (z-1)^2 + 1$ implies that the locus related to z will likely exhibit symmetry around the point $z = 1$. Identifying and using this symmetry can simplify analysis and sketching.

Basic Foundational Theory

- **Complex Arithmetic**: $z = x+yi$, $z^2 = (x^2-y^2)+2xyi$. Modulus $|z| = \sqrt{x^2+y^2}$.
- **Geometric Transformations**:
  - $z \mapsto z+c$: Translation by vector representing c.
  - $z \mapsto cz$: Rotation by arg(c) and scaling by $|c|$.
  - $z \mapsto z^2$: If $z = re^{i\theta}$, then $z^2 = r^2e^{i2\theta}$ (squares modulus, doubles argument).
- **Locus**: A set of points satisfying a given condition. $|z-a| = R$ is a circle. More complex conditions lead to other curves.
- **Cassini Ovals**: Locus of points P such that the product of distances from P to two fixed points (foci) is constant. The equation $|z^2-a^2| = b^2$ (or $|(z-c)^2-a^2| = b^2$ after translation) can represent Cassini ovals. Here $|u^2-(-1)| = 2$.

Practice Problems 2

Practice Problem 1: Locus Calculation

Find the locus of $|z^2 - 1| = 1$. [5 marks]

Solution to Practice Problem 1

Let $z = x + yi$. Then $z^2 - 1 = (x^2 - y^2 - 1) + 2xyi$.
Given $|z^2 - 1| = 1$, so $|(x^2 - y^2 - 1) + 2xyi|^2 = 1^2$.
$(x^2 - y^2 - 1)^2 + (2xy)^2 = 1$.
This is the Cartesian equation of the locus.
In polar coordinates, let $z = re^{i\theta}$. Then $z^2 = r^2 e^{i2\theta}$.
The equation $|z^2 - 1| = 1$ is the definition of a Lemniscate of Bernoulli with foci at ± 1.
Expanding: $x^4 + y^4 + 1 - 2x^2y^2 - 2x^2 + 2y^2 + 4x^2y^2 = 1$.
$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.
Or, in polar coordinates $(r^2)^2 - 2(r^2 \cos(2\theta)) = 0 \implies r^4 - 2r^2 \cos(2\theta) = 0$.
Since $r \neq 0$ generally for the curve, $r^2 = 2\cos(2\theta)$.

$$\boxed{(x^2 + y^2)^2 = 2(x^2 - y^2) \quad \text{or} \quad r^2 = 2\cos(2\theta) \text{ (Lemniscate of Bernoulli)}}$$

Practice Problem 2: Minimum Modulus

Determine the minimum $|z^2 + z|$ and the corresponding value(s) of z. [4 marks]

Solution to Practice Problem 2

We want to minimize $|z^2 + z| = |z(z + 1)|$. The modulus of a product is the product of the moduli: $|z(z + 1)| = |z||z + 1|$.
This expression is zero if either $|z| = 0$ or $|z + 1| = 0$.
If $|z| = 0$, then $z = 0$. In this case, $|z^2 + z| = |0| = 0$.
If $|z + 1| = 0$, then $z + 1 = 0 \implies z = -1$. In this case, $|z^2 + z| = |(-1)^2 + (-1)| = |1 - 1| = |0| = 0$.
Since the modulus is always non-negative ($|f(z)| \geq 0$), the minimum possible value is 0.

$$\boxed{\text{Minimum } |z^2+z| = 0 \text{ at } z = 0 \text{ or } z = -1}$$

Further Problems 2

Further Problem 1: Advanced Locus

Find the locus of $|z^2 - 2z + 3| = 2$. [5 marks]

Solution to Further Problem 1

Let $w = z^2-2z+3$. We can complete the square for z: $w = (z^2-2z+1)+2 = (z-1)^2+2$.
We are given $|w| = 2$, so $|(z-1)^2 + 2| = 2$.
Let $u = z - 1$. Then $|u^2 + 2| = 2$.
Let $u = p + qi$. Then $u^2 = p^2 - q^2 + 2pqi$.
So, $|(p^2 - q^2 + 2) + 2pqi| = 2$.
Squaring both sides: $(p^2 - q^2 + 2)^2 + (2pq)^2 = 4$.
$(p^2 - q^2)^2 + 4(p^2 - q^2) + 4 + 4p^2q^2 = 4$.
$(p^2 - q^2)^2 + 4p^2q^2 + 4(p^2 - q^2) = 0$.
$(p^2 + q^2)^2 + 4(p^2 - q^2) = 0$.
Let $p = (x - 1)$ and $q = y$.

$$((x-1)^2 + y^2)^2 + 4((x-1)^2 - y^2) = 0$$

In polar coordinates for $u = Re^{i\theta}$: $(R^2)^2+4(R^2\cos(2\theta)) = 0 \implies R^4+4R^2\cos(2\theta) = 0$.
Since $R = |u| = |z-1|$, $R \geq 0$.
So either $R^2 = 0$ (which means $u = 0 \implies z = 1$) or $R^2 = -4\cos(2\theta)$.
For R^2 to be non-negative, we need $\cos(2\theta) \leq 0$. This describes a Lemniscate of Gerono (figure-eight curve) centered at $u = 0$ (i.e., $z = 1$).

$$\boxed{((x-1)^2 + y^2)^2 + 4((x-1)^2 - y^2) = 0 \quad \text{or} \quad R^2 = -4\cos(2\theta) \text{ where } R = |z-1|}$$

Further Problem 2: Geometric Transformation

Analyze the transformation $w = z^2 + z$ geometrically. [6 marks]

Solution to Further Problem 2

The transformation is $w = z^2 + z$.
We can complete the square: $w = (z^2 + z + \frac{1}{4}) - \frac{1}{4} = (z + \frac{1}{2})^2 - \frac{1}{4}$.
This can be seen as a sequence of transformations: 1. $u_1 = z + \frac{1}{2}$: Translation of z by $+\frac{1}{2}$ (half unit to the right).
2. $u_2 = u_1^2 = (z + \frac{1}{2})^2$: Squaring. This maps circles centered at the origin in the u_1-plane to circles in the u_2-plane, and lines through the origin in the u_1-plane to rays from the origin in the u_2-plane.
3. $w = u_2 - \frac{1}{4} = (z + \frac{1}{2})^2 - \frac{1}{4}$: Translation by $-\frac{1}{4}$ (quarter unit to the left).

Alternatively, $w = z(z + 1)$.
This involves multiplying z by $z + 1$.
If $z = re^{i\theta}$ and $z + 1 = r_1 e^{i\phi_1}$, then $w = rr_1 e^{i(\theta+\phi_1)}$.
The modulus of w is the product of the modulus of z and the modulus of $z + 1$.
The argument of w is the sum of the argument of z and the argument of $z + 1$.
This transformation maps the imaginary axis ($z = iy$) to $w = (iy)^2 + iy = -y^2 + iy$.
This is a parabola $x = -y^2$ opening to the left, but with x as the real part and y as the imaginary part, so $Re(w) = -(Im(w))^2$.
It maps the unit circle $|z| = 1$ to a cardioid-like shape.

$\boxed{w = (z + 1/2)^2 - 1/4\text{: Transl by +1/2, Square, Transl by -1/4.}}$

$\boxed{w = z(z + 1)\text{: Product transform}}$

Conclusion: Your Path to Mathematical Mastery

This guide has provided you with a powerful toolset for tackling IB Math AA HL Paper 3 challenges. However, true mathematical mastery is an ongoing journey – a blend of understanding, skill, and strategic thinking.

All solutions and commentary are original work by Rishabh Kumar, Mathematics Elevate Academy.

Key Takeaways for Exam Success:

- **Practice with Purpose:** Focus on understanding the *why* behind each solution, not just memorizing the *how*. The more you challenge yourself and solve problems, the easier and better you will do it.
- **Embrace Your Mistakes:** Every mistake is an opportunity to learn. Analyze what worked and what you can improve next time.
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