International Baccalaureate Diploma Programme Mathematics Analysis and Approaches Higher Level

Master Implicit Differentiation

Unlock 7-Scorer Potential

Concept | Practice Set | Challenge Set | April 2025 Edition

Mathematics Elevate Academy

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Rishabh Kumar

Elite Private Mentor for IB Math AA/AI HL

Founder, Mathematics Elevate Academy

Alumnus of IIT Guwahati & Indian Statistical Institute

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www.mathematicselevateacademy.com

www.linkedin.com/in/rishabh-kumar-iitg-isi/

Disclaimer

Preface

*This guide is designed for ambitious learners aiming for excellence—those who wish to truly **understand**, not merely memorize.* Crafted with precision and clarity, the content within reflects the standards of elite mathematical training. Whether you're preparing for your IB examinations or seeking to deepen your foundation in proof-based mathematics, this resource offers a step-by-step exploration, richly illustrated with:

- Carefully structured **conceptual breakdowns**
- **Solved examples** with insightful commentary
- Handpicked **IB exam style problems**
- A thoughtfully curated set of **practice and challenge problems**

As a global educator specializing in advanced mathematics and statistical thinking, I am tailored to mentor students across continents to achieve the coveted **Level 7** in IB Math AA HL. This note is a distilled reflection of that experience—crafted not just to teach, but to *inspire*.

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Introduction

Unlock your mathematical potential with **Mathematics Elevate Academy's** master selected topics series, crafted for ambitious IB DP Mathematics AA HL students.

Mathematics, at its highest level, is not merely a subject of numbers—it's a disciplined language of logic, structure, and proof. Among the most elegant tools **Implicit Differentiation**—to differentiate an implicit function.

In the **IB Mathematics: Analysis and Approaches HL** curriculum, **Implicit Differentiation** stands as a gateway to advanced mathematical thinking. It is a cornerstone concept, often feared by students due to its abstract nature, yet revered for its logical beauty and power. Mastering this technique not only enhances problem-solving ability but cultivates a mindset essential for mathematical maturity and success in IB Exam-style reasoning.

Implicit Differentiation

A Comprehensive Guide for IB Math AA HL

In many real-world scenarios, functions are not given explicitly (like $y = f(x)$), but rather as equations involving both x and y together — for example, $x^2 + y^2 = 25$. To find $\frac{dy}{dx}$, we use **implicit differentiation**.

When to Use Implicit Differentiation

Use implicit differentiation when:

- y is not isolated (i.e., not written as $y = f(x)$)
- The equation involves both x and y and is difficult or impossible to solve for y
- There are higher powers or trigonometric/logarithmic functions of y

Basic Principle

Differentiate both sides of the equation with respect to x, treating y as a function of x.

Always apply the **chain rule** when differentiating any term involving y.

Step-by-Step Procedure

1. Differentiate both sides with respect to x
2. Apply the chain rule wherever y appears:

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

3. Collect all $\frac{dy}{dx}$ terms on one side
4. Factor out $\frac{dy}{dx}$
5. Solve for $\frac{dy}{dx}$

Examples

Example 1: Circle Equation

Given:

$$x^2 + y^2 = 25$$

Differentiate both sides:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$2y \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Example 2: Product of x and y

Given:

$$xy = 1$$

Use product rule:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$x \cdot \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Example 3: Trigonometric Equation

Given:

$$\sin(xy) = x + y$$

Differentiate both sides:

$$\cos(xy) \cdot \left(y + x \cdot \frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

Solve algebraically for $\frac{dy}{dx}$.

Advanced Example: $x^2 + xy + y^2 = 7$

Given:

$$x^2 + xy + y^2 = 7$$

Differentiate both sides:

$$2x + x \cdot \frac{dy}{dx} + y + 2y \cdot \frac{dy}{dx} = 0$$

Group terms:

$$(x + 2y)\frac{dy}{dx} = -2x - y \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

General Method for $I(x, y) = 0$

(a) General Method

To find $\frac{dy}{dx}$ for implicit functions, differentiate each term with respect to x, treating y as a function of x. Then, collect all terms involving $\frac{dy}{dx}$ on one side and solve.

(b) Partial Derivative Shortcut Formula

For an implicit function $I(x, y) = 0$, the derivative can also be found using:

$$\frac{dy}{dx} = -\frac{\frac{\partial I}{\partial x}}{\frac{\partial I}{\partial y}}$$

Here:

- $\frac{\partial I}{\partial x}$ is the partial derivative of $I(x, y)$ with respect to x, treating y as constant

- $\dfrac{\partial I}{\partial y}$ is the partial derivative of $I(x, y)$ with respect to y, treating x as constant

(c) Form of the Solution

In the case of implicit functions, generally, both x and y appear in the answer of $\dfrac{dy}{dx}$.

Additional Example 1: $xy + y^x = 2$

Let:

$$u = xy, \quad v = y^x \quad \Rightarrow \quad u + v = 2$$

Differentiate both sides with respect to x:

$$\frac{du}{dx} + \frac{dv}{dx} = 0$$

Now:

$$u = xy \Rightarrow \frac{du}{dx} = y + x \cdot \frac{dy}{dx}$$

For $v = y^x$, use logarithmic differentiation:

$$v = y^x \Rightarrow \ln v = x \ln y \Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \ln y + \frac{x}{y} \cdot \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = y^x \left(\ln y + \frac{x}{y} \cdot \frac{dy}{dx} \right)$$

Substitute back:

$$\frac{du}{dx} + \frac{dv}{dx} = y + x \cdot \frac{dy}{dx} + y^x \left(\ln y + \frac{x}{y} \cdot \frac{dy}{dx} \right) = 0$$

Group terms with $\frac{dy}{dx}$:

$$\left(x + \frac{xy^x}{y} \right) \cdot \frac{dy}{dx} = -y - y^x \ln y \Rightarrow \frac{dy}{dx} = \frac{-y - y^x \ln y}{x + \dfrac{xy^x}{y}}$$

Alternative Method (Using Partial Derivatives)

Let:

$$I(x,y) = xy + y^x - 2$$

Then:

$$\frac{dy}{dx} = -\frac{\frac{\partial I}{\partial x}}{\frac{\partial I}{\partial y}}$$

Compute the partial derivatives:

$$\frac{\partial I}{\partial x} = y + y^x \ln y, \quad \frac{\partial I}{\partial y} = x + \frac{xy^x}{y}$$

So:

$$\frac{dy}{dx} = -\frac{y + y^x \ln y}{x + \frac{xy^x}{y}}$$

Ans. $\boxed{\frac{dy}{dx} = \frac{-y - y^x \ln y}{x + \frac{xy^x}{y}}}$

Additional Example 2: Recursive Function $y = \frac{\sin x}{1+\frac{\cos x}{1+\frac{\sin x}{1+\frac{1}{1+\cos x}}}}$

Prove that:

$$\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}.$$

Solution (Method 1): Using Simplification and Differentiation

Given function is:

$$y = \frac{\sin x}{1+\frac{\cos x}{1+\frac{\sin x}{1+\frac{1}{1+\cos x}}}}.$$

Assume:

$$y = \frac{\sin x}{1+\frac{\cos x}{1+y}}.$$

Simplify:

$$y = \frac{\sin x(1+y)}{1+y\cos x}.$$

Cross-multiplying:

$$y(1+y\cos x) = (1+y)\sin x.$$

Expand and rewrite:

$$y + y^2\cos x - y\sin x - \sin x = 0.$$

Differentiate both sides:

$$\frac{d}{dx}\left(y + y^2 + y\cos x\right) = \frac{d}{dx}\left((1+y)\sin x\right)$$

Apply product/chain rules:

$$(1 + 2y + \cos x - \sin x)\cdot\frac{dy}{dx} = (1+y)\cos x + y\sin x$$

Thus:

$$\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$$

Ans. $\boxed{\dfrac{dy}{dx} = \dfrac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}}$

Tips and Tricks

- Apply the chain rule wherever y appears
- Keep expressions neat — always write $\frac{dy}{dx}$ explicitly
- Collect and factor terms carefully to isolate $\frac{dy}{dx}$

Applications

- Finding the slope of a tangent to curves not in explicit form
- Analyzing level curves in multivariable calculus

- Related rates in geometry and physics

Practice Problems with Solutions

Problem 1: $x + y = \sin(x - y)$

Differentiate both sides:

$$1 + \frac{dy}{dx} = \cos(x - y)\left(1 - \frac{dy}{dx}\right)$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\cos(x - y) - 1}{1 + \cos(x - y)}$$

Ans. $\boxed{\frac{dy}{dx} = \frac{\cos(x - y) - 1}{1 + \cos(x - y)}}$

Problem 2: $x^2 + xe^y + y = 0$

Differentiate:

$$2x + e^y + xe^y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-(2x + e^y)}{xe^y + 1}$$

At $(0, 0)$:

$$\left.\frac{dy}{dx}\right|_{(0,0)} = -1$$

Ans.

$$\boxed{\frac{dy}{dx} = \frac{-(2x + e^y)}{xe^y + 1}, \quad \left.\frac{dy}{dx}\right|_{(0,0)} = -1}$$

Practice Problems

1. $x^3 + y^3 = 6xy$
2. $e^{xy} = x + y$

3. $\tan(x + y) = x^2 + y^2$
4. $\ln(xy) + x = y$
5. $\sin(xy) + y = x^2$
6. Find $\frac{dy}{dx}$:
   (a) $\sin y + x^2 + 4y = \cos x$
   (b) $3xy^2 + \cos y^2 = 2x^3 + 5$
   (c) $5x^2 - x^3 \sin y + 5xy = 10$
   (d) $x - \cos x^2 + \frac{y^2}{x} + 3x^5 = 4x^3$
   (e) $\tan(5y) - y \sin x + 3xy^2 = 9$

IB Exam-Style Problems on Implicit Differentiation

1. **[6 marks]** The curve is defined by the equation:

$$x^2 + xy + y^2 = 7.$$

   (a) Use implicit differentiation to find $\dfrac{dy}{dx}$.
   (b) Hence, find the equation of the tangent to the curve at the point $(1, 2)$.

2. **[5 marks]** The variables x and y satisfy the equation:

$$\sin(xy) = x + y.$$

   Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of x and y.

3. **[6 marks]** Let the curve be defined implicitly by:

$$e^{xy} + x^2 = y.$$

   (a) Find $\dfrac{dy}{dx}$.

(b) Find the value of $\frac{dy}{dx}$ at the point $(0, 1)$.

4. **[7 marks]** The function is defined implicitly by:

$$x^2y + y^3 = 6.$$

(a) Find $\frac{dy}{dx}$.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

5. **[4 marks]** A curve is defined implicitly by:

$$\ln(y) + yx = x^2.$$

Find the value of $\frac{dy}{dx}$ at the point where $x = 1$ and $y = 1$.

6. **[5 marks]** Given the implicit equation:

$$x^2 + y^2 = \tan^{-1}(xy),$$

differentiate both sides to find $\frac{dy}{dx}$ in terms of x and y.

7. **[6 marks]** Let $x = y^y$. Find $\frac{dy}{dx}$ using logarithmic differentiation.

Challenging Olympiad Problems on Implicit Differentiation

1. Let the function $f(x, y)$ be implicitly defined by the relation

$$x^2 + y^2 + \ln(xy) = 1, \quad (x > 0, y > 0).$$

Find $\frac{dy}{dx}$ in terms of x and y.

2. The curve C is defined implicitly by the equation

$$x^y + y^x = 2.$$

Find $\frac{dy}{dx}$ at the point where $x = 1$ and $y = 1$.

3. Let $x^3 + y^3 + 3axy = 0$ be the equation of a curve. Find the coordinates of the point(s) on the curve where the tangent is horizontal.

4. A curve is defined implicitly by

$$x^2y + y^2x = 6.$$

Find $\frac{d^2y}{dx^2}$ in terms of x and y.

5. Let $x^2 + y^2 + z^2 = xyz$, where z is a function of x and y. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

6. Consider the curve given implicitly by

$$\sin(xy) + \cos(yx) = x^2 + y^2.$$

Find $\frac{dy}{dx}$ at the point $(0, 0)$.

7. Let $x = \tan(yx)$, where y is a function of x. Find $\frac{dy}{dx}$ in terms of x and y.

Summary

Implicit differentiation is used when the relationship between x and y is not explicitly defined. The key steps are:

- Differentiate both sides with respect to x
- Apply the chain rule when differentiating y

- Solve algebraically for $\frac{dy}{dx}$

Pro tip: Always treat y as a function of x, even when it doesn't look like one!

Conclusion: Your Path to Mathematical Mastery

This guide has provided you with a powerful toolset for mastering implicit differentiation. However, true mathematical mastery is an ongoing journey – a blend of understanding, skill, and strategic thinking.

Key Takeaways for Exam Success:

- **Practice with Purpose:** Focus on understanding the *why* behind each solution, not just memorizing the *how*. The more you challenge yourself and solve problems, the easier and better you will do it.
- **Embrace Your Mistakes:** Every mistake is an opportunity to learn. Analyze what worked and what you can improve next time.
- **Time is a Crucial Asset** Simulate the exam and prepare well to achieve success.

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