# International Baccalaureate Diploma Programme Mathematics Analysis and Approaches Higher Level

## Paper 3 Elite Edition

Unlock 7-Scorer Potential

Exclusive Exam-Style Problems | Expert Strategies | April 2025 Edition


**Mathematics Elevate Academy**

*Excellence in Advanced Math Education*

**Rishabh Kumar**

**Founder, Mathematics Elevate Academy**

**Elite Private Mentor for IB Math HL**

*Alumnus of IIT Guwahati & Indian Statistical Institute*

*5+ Years of Teaching Experience*

# Disclaimer

# Introduction

Unlock your mathematical potential with **Mathematics Elevate Academy's** exclusive Paper 3 problem set, crafted for ambitious IB DP Mathematics AA HL students.

This collection provides a *rigorous and enriching* preparation experience tailored for the current syllabus (2021 examinations onward).

This guide empowers you to:

- **Master Elite-Level Challenges:** Enhance your depth of understanding with questions that go beyond the textbook.
- **Understand the IB Marking Scheme:** Step-by-step examiner-style solutions show how to score full marks.
- **Avoid Hidden Pitfalls:** Efficient strategies and structured thinking save time under pressure.
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# Problem 1: Exploring a Family of Functions (Maximum Points: 25)

**Consider the following family of functions:**

$$f_n(x) = e^{-2x}(2x)^n, \quad \text{where } x \geq 0 \text{ and } n \in \mathbb{Z}^+.$$

## Problem 1.1 (4 Points)

**Properties of $f_n(x)$:**

- Sketch the graph of $y = f_1(x)$ (i.e., take $n = 1$) and determine the coordinates of its maximum point.

## Problem 1.2 (6 Points)

**Area Under the Curve:**

- Prove that the area enclosed by the curve $y = f_1(x)$, the x-axis, and the vertical line $x = b$ (where $b > 0$) is given by:

$$\text{Area} = \frac{1 - e^{-2b}(2b+1)}{2}.$$

## Problem 1.3 (2+1 = 3 Points)

**Total Area Under the Curve & Limit Computation:**

- Use L'Hôpital's Rule to compute:

$$\lim_{b\to\infty} \frac{\left(1 - e^{-2b}(2b+1)\right)}{2}.$$

- Using the result from the above part, find the value of $T_1$.

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**Problem Statement:**

Suppose, we define the total area $T_n$ between the curve $y = f_n(x)$ and the x-axis as:

$$T_n = \int_0^\infty f_n(x)\,dx,$$

which can also be expressed using a limit:

$$T_n = \lim_{b\to\infty} \int_0^b f_n(x)\,dx.$$

## Problem 1.4 (2+1 = 3 Points)

**Numerical Integration:**

- Given that $T_2 = 1$ and $T_3 = 3$, use a graphic display calculator to determine:
  (a) $T_4$
  (b) $T_5$

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## Problem 1.5 (1 Point)

**Pattern Recognition:**

- Based on the results obtained, suggest a general formula for $T_n$ in terms of $n$, where $n$ is a positive integer.

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## Problem 1.6 (8 Points)

**Proof by Induction:**

- Prove the formula you proposed in Problem 1.5 using mathematical induction. Assume that for any positive integer $n$:

$$\lim_{x \to \infty} (2x)^n e^{-2x} = 0$$

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# Problem 2: Exploring the Maximum Product of Positive Real Numbers (Maximum Points: 30)

Let $y_1, y_2 \in \mathbb{R}^+$, such that $y_1 + y_2 = 10$.

## Problem 2.1 (2 Points)

**Expressing the Product as a Function:**

- Express the product of $y_1$ and $y_2$ as a function, $f$, in terms of $y_1$ only.

## Problem 2.2 (1+1 = 2 Points)

**Maximizing the Product:**

- Determine the value of $y_1$ that maximizes the function.
- Show that the maximum product of $y_1$ and $y_2$ is 25.

**Problem Statement:** Define $M_n(S)$ as the maximum product of $n$ positive real numbers whose sum is $S$, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$. For $n = 2$, the maximum product is given by:

$$M_2(S) = \left(\frac{S}{2}\right)^2.$$

## Problem 2.3 (1 Point)

**Generalizing the Maximum Product:**

- Verify that the formula $M_2(S) = \left(\frac{S}{2}\right)^2$ holds true for $S = 10$.

**Problem Statement:** Consider $n$ positive real numbers $y_1, y_2, \ldots, y_n$. The geometric mean of these numbers is defined as:

$$(y_1 y_2 \cdots y_n)^{\frac{1}{n}}.$$

It is known that the geometric mean is always less than or equal to the arithmetic mean, so:

$$(y_1 y_2 \cdots y_n)^{\frac{1}{n}} \leq \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

## Problem 2.4 (2+4 = 6 Points)

**Geometric and Arithmetic Means:**

- Prove that the geometric mean equals the arithmetic mean when $y_1 = y_2 = \cdots = y_n$.
- Use this result to derive the formula:

$$M_n(S) = \left(\frac{S}{n}\right)^n.$$

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## Problem 2.5 (1+1+1 = 3 Points)

**Calculating Specific Values of $M_n(S)$:**

1. $M_3(10)$
2. $M_4(10)$
3. $M_5(10)$

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**Problem Statement:** For $n \in \mathbb{Z}^+$, let $P(S)$ represent the maximum value of $M_n(S)$ across all possible values of $n$.

## Problem 2.6 (2 Points)

**Maximum Value of $M_n(S)$:**

- Write down the value of $P(10)$ and the corresponding value of $n$.

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## Problem 2.7 (3 Points)

**Maximum Product for $S = 20$:**

- Determine the value of $P(20)$ and the corresponding value of $n$.

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**Problem Statement:** Define the function $g$ as:

$$\ln(g(y)) = y \ln\left(\frac{S}{y}\right),$$

where $y \in \mathbb{R}^+$. The graph of $z = g(y)$ has a point $A$ representing the maximum point on the graph.

## Problem 2.8 (6 Points)

**Function $g(y)$ and Its Maximum Point:**

- Find the $y$-coordinate of point $A$ in terms of $S$.

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## Problem 2.9 (2 Points)

**Verifying the Relationship Between $g(y)$ and $M_y(S)$:**

- Show that $g(y) = M_y(S)$, where $y \in \mathbb{Z}^+$.

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## Problem 2.10 (3 Points)

**Largest Product for $S = 100$:**

- Using your result from Problem 2.8, calculate the largest possible product of positive numbers whose sum is 100. Express your answer in the form $a \times 10^n$, where $1 \leq a < 10$ and $n \in \mathbb{Z}^+$.

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## Problem 3: Exploring Intersection Points Between Graphs (Maximum Points: 24)

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**This question explores the intersection points between the graph of $y = \log_a(2x)$, where $a \in \mathbb{R}^+$, $a \neq 1$, and the line $y = 2x$.**

You may use either the change of base formula:

$$\log_a(2x) = \frac{\ln(2x)}{\ln(a)}$$

or a graphic display calculator with a "logarithm to any base" feature.

The function $f$ is defined as:

$$f(2x) = \log_a(2x), \quad \text{where } 2x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+,\ a \neq 1.$$

### Problem 3.1 (4 Points)

**Plotting Graphs:**

- Plot the following three graphs on the same set of axes:

$$y = \log_2(2x), \quad y = \log_{10}(2x), \quad y = 2x$$

  Clearly label each graph with its equation and identify the value of any non-zero x-axis intercepts.

### Problem 3.2 (5 Points)

**Minimum Value Analysis:**

- For $a = e$, use calculus to determine the minimum value of:

$$2x - \ln(2x)$$

and justify that this value is indeed a minimum.

## Problem 3.3 (1 Point)

**Inequality Deduction:**

- Using the result from Problem 3.2, deduce that:

$$2x > \ln(2x)$$

## Problem 3.4 (4 Points)

**Intersection Analysis:**

- Complete the following table by analyzing the graph of $y = \log_a(2x)$ for different values of $a$:

| Interval | Number of intersection points |
|---|---|
| $0 < a < 1$ | $p$ |
| $1 < a < 1.4$ | $q$ |
| $1.5 < a < 2$ | $r$ |

## Problem 3.5 (8 Points)

**Tangency Point:**

- For $1.4 \leq a \leq 1.5$, there exists a value of $a$ such that the line $y = 2x$ is tangent to the graph of $y = \log_a(2x)$ at a point $P$.
- Find the exact coordinates of $P$ and the exact value of $a$.

## Problem 3.6 (1+1 = 2 Points)

**Intersection Scenarios:**

- Write down the exact set of values for $a$ such that the graphs of $y = \log_a(2x)$ and $y = 2x$ have:
  (a) two intersection points
  (b) no intersection points

# Problem 4: Exploring Linear and Quadratic Functions Using Arithmetic Sequences (Maximum Points: 31)

**Let $L(x) = mx + c$, where $x \in \mathbb{R}$, $m, c \in \mathbb{R}$, and $m, c \neq 0$.**

Suppose $r \in \mathbb{R}$ is the root of $L(x) = 0$. If $m, r$, and $c$, in that order, form an arithmetic sequence, then $L(x)$ is called an AS-linear function.

## Problem 4.1 (2 Points)

**Verifying an AS-Linear Function:**

- Prove that $L(x) = 4x - \frac{8}{3}$ is an AS-linear function.

## Problem 4.2 (1+4+1 = 6 Points)

**Properties of AS-Linear Functions:**

- Consider $L(x) = mx + c$ and show that:

(i) $r = -\frac{c}{m}$

(ii) If $L(x)$ is an AS-linear function, then:

$$L(x) = m\left(x - \frac{m}{2}\right) + \frac{m}{2}$$

(iii) State any additional restrictions on the value of $m$

## Problem 4.3 (3 Points)

**Finding Integer AS-Linear Functions:**

- There are exactly three sets of integer values for $m, r$, and $c$ that form an AS-linear function. One of these is $L(x) = -2x - 4$. Use the result from Problem 4.2 to find the other two AS-linear functions with integer values of $m, r,$ and $c$.

**Consider the quadratic function** $Q(x) = ax^2 + bx + c$**, where** $x \in \mathbb{R}$**,** $a \in \mathbb{R}$**,** $a \neq 0$**, and** $b, c \in \mathbb{R}$**.**

Let $r_1, r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

## Problem 4.4 (1+1 = 2 Points)

**Properties of Quadratic Functions:**

- Express the following in terms of the given variables:

(i) Write an expression for the sum of the roots, $r_1 + r_2$, in terms of $a$ and $b$

(ii) Write an expression for the product of the roots, $r_1 r_2$, in terms of $a$ and $c$

**Consider the quadratic function** $Q(x) = px^2 + qx + r$**, where** $x \in \mathbb{R}$**,** $p \in \mathbb{R}$**,** $p \neq 0$**, and** $q, r \in \mathbb{R}$**.**

If $p, r_1, q, r_2,$ and $r$, in that order, form an arithmetic sequence, where $r_1$ and $r_2$ are the roots of $Q(x) = 0$, then $Q(x)$ is said to be an AS-quadratic function.

## Problem 4.5 (1+2+3 = 6 Points)

**Properties of AS-Quadratic Functions:**

- Given that $Q(x)$ is an AS-quadratic function:

(i) Write an expression for $r_2 - r_1$ in terms of $p$ and $q$

(ii) Use your answers to Problems 4.4(i) and 4.5(i) to show that:

$$r_1 = \frac{-q - \sqrt{q^2 - 4pr}}{2p}$$

(iii) Use the result from Problem 4.5(ii) to show that $q = 0$ or $p = -\frac{1}{2}$

## Problem 4.6 (5 Points)

**AS-Quadratic Functions When $q = 0$:**

- Determine the two AS-quadratic functions that satisfy this condition.

## Problem 4.7 (2+5 = 7 Points)

**AS-Quadratic Functions When $p = -\frac{1}{2}$:**

- Consider the case where $p = -\frac{1}{2}$:

(i) Find an expression for $r_1$ in terms of $q$

(ii) Hence, or otherwise, determine the exact values of $q$ and $r$ such that AS-quadratic functions are formed. Give your answers in the form:

$$-\frac{p \pm q\sqrt{s}}{2}$$

where $p, q, s \in \mathbb{Z}^+$

# Problem 5: Exploring Polygonal Numbers and Their Properties

**(Maximum Points: 27)**

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**This task explores polygonal numbers—numbers that can be visually arranged in the shape of regular polygons like triangles, squares, pentagons, etc.**

A number is said to be polygonal if it can be formed by evenly arranging dots into the shape of a regular polygon. The $k$-gonal number sequence is defined by the formula:

$$P_k(n) = \frac{(k-2)n^2 - (k-4)n}{2}$$

where $n \in \mathbb{Z}^+$, $k \geq 3$.

## Problem 5.1 (2+2 = 4 Points)

**Triangular Numbers:**

(i) Verify the formula for triangular numbers (i.e., when $k = 3$) and simplify.

(ii) Determine the position $n$ of the triangular number equal to 406.

## Problem 5.2 (2+1+1 = 4 Points)

**Geometric Identity with Triangular Numbers:**

(i) Prove that the sum of two consecutive triangular numbers satisfies:

$$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = (n+1)^2 + (n+1)$$

Then simplify and compare the result with a perfect square.

(ii) Explain geometrically how stacking two triangular layers relates to forming a square-like arrangement.

(iii) Draw a diagram showing how this works for $n = 3$.

## Problem 5.3 (3 Points)

**A Surprising Square:**

- Show that the expression $8T_n + 1$ always produces the square of an odd number.

## Problem 5.4 (3 Points)

**Discovering Hexagonal Numbers:**

- The $n$th hexagonal number can be defined using a linear progression:

$$H(n) = 1 + 5 + 9 + 13 + \cdots + (4n - 3)$$

  Derive a closed-form expression for the $n$th hexagonal number based on the above sequence.

## Problem 5.5 (5 Points)

**Shared Values in Two Sequences:**

- Find the smallest integer greater than 1 that appears in both the triangular number and hexagonal number sequences.

## Problem 5.6 (8 Points)

**Induction on Polygonal Formula:**

- Prove using mathematical induction that for any $k \in \mathbb{Z}^+, k \geq 3$, the number of dots forming a $k$-gon with $n$ layers is:

$$P_k(n) = \sum_{i=1}^{n} [1 + (k-2)(i-1)] = \frac{(k-2)n^2 - (k-4)n}{2}$$

## Problem 6: Exploring the Geometry of Cubic Polynomials (Maximum Points: 28)

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**Consider a special form of cubic polynomial with one real and two non-real complex conjugate roots:**

$$f(x) = (x - r)(x^2 - 2ax + a^2 + b^2), \quad x \in \mathbb{R}$$

and the corresponding equation over the complex numbers:

$$(z - r)(z^2 - 2az + a^2 + b^2) = 0, \quad z \in \mathbb{C}$$

For the first few parts, let $r = 1$, $a = 4$, and $b = 1$.

### Problem 6.1 (1+1 = 2 Points)

**Roots and Basic Properties:**

- The equation becomes:

$$(z - 1)(z^2 - 8z + 17) = 0$$

(i) Given that two of the solutions are $z = 1$ and $z = 4 + i$, determine the third solution.

(ii) Show that the average (mean) of the complex pair of roots is a real number, and state its value.

### Problem 6.2 (4 Points)

**Tangent Line to a Cubic Curve:**

- Let $f(x) = (x - 1)(x^2 - 8x + 17)$, $x \in \mathbb{R}$. Prove that the straight line $y = x - 1$ is tangent to the curve $y = f(x)$ at the point $A = (4, 3)$.

## Problem 6.3 (2 Points)

**Sketching the Graph:**

- Sketch the graph of $y = f(x)$. Clearly show:
  - the real root
  - the point of tangency at $A$
  - the point where the tangent line intersects the x-axis

## Problem 6.4 (2+6 = 8 Points)

**General Form - Derivative and Tangent Geometry:**

- Let $g(x) = (x - r)(x^2 - 2ax + a^2 + b^2)$, where $r, a, b \in \mathbb{R}$ and $b > 0$.

(i) Show that the derivative is:

$$g'(x) = 2(x - r)(x - a) + (x^2 - 2ax + a^2 + b^2)$$

(ii) Show that the tangent to the curve at the point $A = (a, g(a))$ intersects the x-axis exactly at $x = r$.

## Problem 6.5 (1 Point)

**Complex Root Interpretation:**

- Using the result of Problem 6.4(i), explain why the non-real roots can be interpreted as complex numbers of the form $a \pm i g'(a)$.

## Problem 6.6 (4+1 = 5 Points)

**Geometry of Roots:**

- Consider $f(x) = (x - r)(x^2 - 2ax + a^2 + 16)$ with point $A = (a, 80)$ and $R = (-2, 0)$.

(i) Determine:

- the value of $a$ and $r$
- the complete set of roots of $(z-r)(z^2-2az+a^2+16)=0$

(ii) State the coordinates of point $C_2$, representing the complex root with negative imaginary part.

## Problem 6.7 (2+1 = 3 Points)

**Inflection Point of the Cubic Curve:**

- Consider $f(x)=(x-r)(x^2-2ax+a^2+b^2)$, where $a \neq r$ and $b>0$.

(i) Show that the x-coordinate of the inflection point $P$ is $\frac{1}{3}(2a+r)$.

(ii) Explain whether point $P$ lies between points $R$ and $A$, to the left of both, or to the right of both.

## Problem 6.8 (2+1 = 3 Points)

**Special Case - Real and Complex Roots Coincide Horizontally:**

- Consider $f(x)=(x-r)(x^2-2rx+r^2+b^2)$ where $a=r$ and $b>0$.

(i) Sketch the graph when $r=1$, $b=2$. Show all important features.

(ii) Express the coordinates of:

- the point of inflection $P$
- the point $A$

in terms of $r$ and $b$.

# Problem 7: Exploring a Family of Curves and Their Properties

**(Maximum Points: 28)**

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**This problem explores a family of curves given by equations of the form:**

$$w^2 = t^3 + At + B, \quad \text{where } A, B \in \mathbb{R}$$

## Problem 7.1 (4 Points)

**Sketching the Graphs:**

- On the same set of axes, sketch the following graphs over the domain $-2 \leq t \leq 2$ and $-2 \leq w \leq 2$. Clearly label the points where each curve intersects the coordinate axes:

(i) $w^2 = t^3$, with $t \geq 0$

(ii) $w^2 = t^3 + 1$, with $t \geq -1$

## Problem 7.2 (1+1 = 2 Points)

**Inflection Points and Curve Analysis:**

- For the curve $w^2 = t^3 + 1$:

(i) Determine the coordinates of the two inflection points.

(ii) By analyzing the curves from Problem 7.1, identify two features that distinguish one from the other.

## Problem 7.3 (2 Points)

**Family of Curves with Parameter $B$:**

- Consider the family of curves:

$$w^2 = t^3 + B, \quad \text{for } t \geq -3,\ B \in \mathbb{Z}^+$$

By changing the value of $B$, suggest two consistent features that all such curves share.

## Problem 7.4 (3+1 = 4 Points)

**Analyzing the Curve $w^2 = t^3 + t$:**

- Consider the curve $w^2 = t^3 + t$, $t > 0$:

(i) Show that the derivative of $w$ with respect to $t$ satisfies:

$$\frac{dw}{dt} = \pm\frac{3t^2 + 1}{2\sqrt{t^3 + t}}$$

(ii) Hence explain why this curve has neither a local maximum nor a local minimum.

## Problem 7.5 (7 Points)

**Points of Inflection:**

- The curve $w^2 = t^3 + t$ has two points of inflection, which occur at symmetric positions about the origin. Determine the $t$-coordinate of the points of inflection in the form:

$$t = \frac{p + q\sqrt{3}}{r}$$

where $p, q, r \in \mathbb{Z}$.

## Problem 7.6 (2+2 = 4 Points)

**Tangent to the Curve at a Rational Point:**

- Define a point $M(t, w)$ as rational if both coordinates are rational numbers. Consider the curve $w^2 = t^3 + 2$, for $t \geq -3$, with rational point $M = (-1, -1)$:

(i) Find the equation of the tangent to the curve at point $M$.

(ii) Determine the coordinates of the second rational point where this tangent intersects the curve.

### Problem 7.7 (5 Points)

**Intersection of Line Segment with the Curve:**

- Let $N = (-1, 1)$, another rational point on the curve $w^2 = t^3 + 2$. Find the coordinates of the point where the line segment $\overline{QN}$ intersects the curve.

# Problem 8: Exploring Properties of Cubic and Quartic Equations (Maximum Points: 28)

**Consider the cubic equation:**

$$z^3 + Az^2 + Bz + C = 0,$$

where $A, B, C \in \mathbb{R}$, and let the roots be $\alpha, \beta, \gamma$.

## Problem 8.1 (3 Points)

**Vieta's Formulas:**

- By expanding $(z-\alpha)(z-\beta)(z-\gamma)$, show that:

$$A = -(\alpha+\beta+\gamma), \quad B = \alpha\beta+\beta\gamma+\gamma\alpha, \quad C = -\alpha\beta\gamma$$

## Problem 8.2 (3+3 = 6 Points)

**Sum of Squares and Differences:**

- Show that:

(i)

$$A^2 - 2B = \alpha^2 + \beta^2 + \gamma^2$$

(ii) Hence prove that:

$$(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 = 2A^2 - 6B$$

## Problem 8.3 (2 Points)

**Complex Roots Condition:**

- Show that if $A^2 < 3B$, then not all roots can be real.

## Problem 8.4 (2 Points)

**Application of Complex Roots Condition:**

- Given the equation $z^3 - 8z^2 + Bz + 1 = 0$, use your result from Problem 8.3 to show that for $B = 17$, the equation must have at least one complex root.

## Problem 8.5 (2+1 = 3 Points)

**Disproving a Conjecture:**

- Noah suggests that if $A^2 \geq 3B$, then all roots are guaranteed to be real. Consider the equation $z^3 - 7z^2 + Bz + 1 = 0$:

(i) Find the smallest positive integer value of $B$ that disproves Noah's claim.

(ii) Explain why this equation always has at least one real root, regardless of the value of $B$.

**Now consider the quartic polynomial:**

$$u^4 + Mu^3 + Nu^2 + Pu + Q = 0$$

with real coefficients and roots $r_1, r_2, r_3, r_4$.

## Problem 8.6 (3+1 = 4 Points)

**Quartic Equation Properties:**

- For the quartic equation:

(i) Find an expression for $r_1^2 + r_2^2 + r_3^2 + r_4^2$ in terms of $M$ and $N$.

(ii) State a condition in terms of $M$ and $N$ that implies the equation must have at least one complex root.

### Problem 8.7 (1 Point)

**Complex Root Analysis:**

- Show that the equation $u^4 - 2u^3 + 3u^2 - 4u + 5 = 0$ must have at least one complex root using your result from Problem 8.6(ii).

### Problem 8.8 (1+1+4 = 6 Points)

**Integer Root and Complex Roots:**

- Consider the equation $u^4 - 9u^3 + 24u^2 + 22u - 12 = 0$ which has at least one integer root:

(i) What does the condition from Problem 8.6(ii) tell you about this equation?

(ii) Identify the integer root of this equation.

(iii) Express the quartic as a product of a linear and a cubic polynomial, and hence show that the equation has at least one complex root.

# Problem 9: Exploring Cubic Functions and Their Properties

**(Maximum Points: 27)**

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**This problem explores the shape and key characteristics of cubic functions of the form:**

$$f(x) = x^3 - 4kx + 3, \quad x \in \mathbb{R}$$

where $k \in \mathbb{R}$ is a constant parameter.

## Problem 9.1 (3+3 = 6 Points)

**Sketching Cubic Functions:**

- On separate coordinate axes, sketch the graph of $y = f(x)$ for the following values of $k$, showing:
  - The y-intercept
  - The coordinates of any points where the gradient is zero

(i) $k = 1$

(ii) $k = 2$

## Problem 9.2 (1 Point)

**First Derivative:**

- Determine the expression for $f'(x)$, the first derivative.

## Problem 9.3 (1+2+1 = 4 Points)

**Critical Points Analysis:**

- Use your result from Problem 9.2 or otherwise, determine the set of values of $k$ for which the function:

(i) Has a point of inflection with a horizontal tangent

(ii) Exhibits both a local maximum and a local minimum

(iii) Has no turning points at all

## Problem 9.4 (3+1 = 4 Points)

**Local Extrema for $k > 0$:**

- Suppose $k > 0$, and the graph has both a maximum and a minimum. Show that:

(i) The y-coordinate of the local maximum is given by:

$$\frac{4}{3}k^{3/2} + 3$$

(ii) The y-coordinate of the local minimum is:

$$-\frac{4}{3}k^{3/2} + 3$$

## Problem 9.5 (2+2+2 = 6 Points)

**Real Roots Analysis:**

- For $k > 0$, find the set of values of $k$ such that the graph of $y = f(x)$ has:

(i) Exactly one real root (i.e., one x-intercept)

(ii) Exactly two real roots

(iii) Exactly three real roots

## Problem 9.6 (6 Points)

**Generalized Cubic Analysis:**

- Consider a more general cubic:

$$g(x) = x^3 - 4kx + h$$

  where $k, h \in \mathbb{R}$.

Determine all the conditions on $k$ and $h$ for which the graph of $y = g(x)$ has exactly one real x-intercept. Justify your answer with reasoning and/or algebra.

## Problem 10: Area Equals Perimeter (Maximum Points: 28)

**This problem explores different geometric shapes in which the numerical value of the area equals the numerical value of the perimeter.**

For instance, a rectangle with dimensions 4 by 5 has an area of 20 and a perimeter of 20. Let the area of a polygon be denoted by $A$ and the perimeter by $P$.

### Problem 10.1 (3 Points)

**Square Condition:**

- Find the side length $s > 0$ of a square such that the area equals the perimeter, i.e., $A = P$.

### Problem 10.2 (1 Point)

**Area of Triangle in Regular Polygon:**

- A regular polygon with $n$ sides can be divided into $n$ identical isosceles triangles. Let the two equal sides of each triangle be of length $r$, and the base of each triangle be $b$. The angle between the equal sides is $\frac{2\pi}{n}$.

- Write an expression for the area of one triangle, $A_T$, in terms of $r$ and $n$.

### Problem 10.3 (2 Points)

**Base of Triangle in Terms of Radius:**

- Using the triangle described in Problem 10.2, show that:

$$b = 2r \sin\left(\frac{\pi}{n}\right)$$

## Problem 10.4 (7 Points)

**Regular Polygon with Area Equal to Perimeter:**

- Assume that a regular polygon with $n$ sides satisfies $A = P$. Using your results from Problems 10.2 and 10.3, show that:

$$A = P \implies \frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right) = 2nr\sin\left(\frac{\pi}{n}\right)$$

and hence deduce:

$$\frac{A}{P} = \frac{r}{4\tan\left(\frac{\pi}{n}\right)}.$$

## Problem 10.5 (3+1 = 4 Points)

**Maclaurin Series and Limit:**

- The Maclaurin series expansion of $\tan(x)$ is given by:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

(i) Use this expansion to evaluate:

$$\lim_{n\to\infty} 4\tan\left(\frac{\pi}{n}\right)$$

(ii) Interpret the result from part (i) in terms of geometry.

## Problem 10.6 (7 Points)

**Right-Angled Triangle – Area Equals Perimeter:**

- Consider a right-angled triangle with sides $a$, $b$, and hypotenuse $\sqrt{a^2+b^2}$, where $a \geq b$. Suppose the area equals the perimeter of the triangle.

(i) Show that the condition becomes:

$$a^2 + b^2 = ab + \frac{1}{2}(a + b)$$

and hence deduce a relation between $a$ and $b$.

## Problem 10.7 (3+1 = 4 Points)

**Integer Solutions:**

- (i) Using the relation found in Problem 10.6, or otherwise, find all integer values of $a$, $b$ such that the triangle has integer sides, area, and perimeter.
- (ii) State the area and perimeter for each of these right-angled triangles.

## Problem 11: Analyzing a Class of Functions (Maximum Points: 31)

**This problem involves analyzing the behavior and notable characteristics of a class of functions defined by:**

$$f_n(x) = x^n(b - x)^n,$$

where $b \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), assume $b = 3$.

### Problem 11.1 (3 Points)

**Graph of $f_1(x)$:**

- Let $f_1(x) = x(3 - x)$. Sketch the graph of $y = f_1(x)$. Identify and label:
  - All x- and y-intercepts.
  - Any local maximum or minimum points, with coordinates.

### Problem 11.2 (6 Points)

**Graph of $f_n(x)$ for $n > 1$:**

- Now consider:

  $$f_n(x) = x^n(3 - x)^n, \quad \text{for } n \in \mathbb{Z}^+,\, n > 1.$$

  Using a graphing tool or calculator, examine the graph of $y = f_n(x)$ for:
  - Odd values: $n = 3$, $n = 5$,
  - Even values: $n = 2$, $n = 4$.

Complete the following table based on your observations:

| Value of $n$ | Local Maxima | Local Minima | Points of Inflexion with Zero Gradient |
|---|---|---|---|
| $n = 3, n = 5$ | | | |
| $n = 2, n = 4$ | | | |

## Problem 11.3 (5 Points)

**Derivative of $f_n(x)$:**

- Show that the derivative of $f_n(x)$ satisfies:

$$f_n'(x) = nx^{n-1}(b - 2x)(b - x)^{n-1}.$$

## Problem 11.4 (2 Points)

**Critical Points of $f_n(x)$:**

- Determine all values of $x$ for which $f_n'(x) = 0$.

## Problem 11.5 (3 Points)

**Point Above the X-Axis:**

- Prove that the point:

$$\left(\frac{b}{2}, f_n\left(\frac{b}{2}\right)\right)$$

lies above the x-axis for all valid values of $n$ and $b \in \mathbb{R}^+$.

## Problem 11.6 (2 Points)

**Positive Derivative at $\frac{b}{4}$:**

- Hence, or otherwise, show that:

$$f_n'\left(\frac{b}{4}\right) > 0$$

  for all $n \in \mathbb{Z}^+$.

## Problem 11.7 (3+2 = 5 Points)

**Behavior at $(0,0)$:**

- Using the derivative test and your result from Problem 11.6, analyze the point $(0,0)$ on the graph of $f_n(x)$:

(i) Prove that it is a local minimum when $n$ is even and $n > 1$, with $b \in \mathbb{R}^+$.

(ii) Show that it is a point of inflexion with zero gradient when $n$ is odd and $n > 1$, again for $b \in \mathbb{R}^+$.

## Problem 11.8 (5 Points)

**Generalized Function $g(x)$:**

- Now consider the more general function:

$$g(x) = x^n(b-x)^{n-m},$$

  where $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$, and $b \in \mathbb{R}^+$.

State the necessary conditions on $n$, $m$, and $b$ such that the equation:

$$x^n(b-x)^n = m$$

has four distinct real solutions for $x$.

# Problem 12: Geometry of the Roots of Unity (Maximum Points: 24)

**Let $z \in \mathbb{C}$, and consider the equation:**

$$z^n = 1.$$

The $n$ complex roots are:

$$z = 1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1},$$

where $\omega = e^{2\pi i/n}$.

Let each root be represented on the Argand plane as points $P_0, P_1, \ldots, P_{n-1}$, lying on the unit circle centered at the origin.

## Problem 12.1 (2+2 = 4 Points)

**Roots of Unity for $n = 3$:**

- Let $n = 3$, and consider the roots $1, \omega, \omega^2$:

(i) Show that:

$$(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1.$$

(ii) Deduce that:

$$\omega^2 + \omega + 1 = 0.$$

## Problem 12.2 (3 Points)

**Geometric Properties for $n = 3$:**

- Let $P_0 = 1$, $P_1 = \omega$, $P_2 = \omega^2$. Show that the product of lengths:

$$|P_0P_1| \cdot |P_0P_2| = 3.$$

## Problem 12.3 (2 Points)

**Roots of Unity for $n = 4$:**

- Let $n = 4$. Factorize $z^4 - 1$ and deduce that:

$$\omega^3 + \omega^2 + \omega + 1 = 0.$$

## Problem 12.4 (4 Points)

**Geometric Properties for $n = 4$:**

- Let $P_0 = 1$, $P_1 = \omega$, $P_2 = \omega^2$, $P_3 = \omega^3$, the fourth roots of unity. Show that:

$$|P_0P_1| \cdot |P_0P_2| \cdot |P_0P_3| = 4.$$

## Problem 12.5 (1 Point)

**General Formula for the Product of Lengths:**

- Based on the above, suggest a formula for the general case:

$$\prod_{k=1}^{n-1} |1 - \omega^k| = ?$$

## Problem 12.6 (1+2 = 3 Points)

**General Expressions for Distances:**

- (i) Express:

$$|P_0P_2| = |1 - \omega^2|, \quad |P_0P_3| = |1 - \omega^3|.$$

- (ii) Write a general expression:

$$|P_0P_{n-1}| = |1-\omega^{n-1}|.$$

## Problem 12.7 (4+3 = 7 Points)

**General Factorization and Product of Lengths:**

- Let:

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

(i) Show that:

$$z^{n-1} + z^{n-2} + \cdots + 1 = \prod_{k=1}^{n-1}(z-\omega^k).$$

(ii) Use this to prove:

$$\prod_{k=1}^{n-1}|1-\omega^k| = n.$$

## Conclusion: Your Path to Mathematical Mastery

This guide has provided you with a powerful toolset for tackling IB Math AA HL Paper 3 challenges. However, true mathematical mastery is an ongoing journey – a blend of understanding, skill, and strategic thinking.

**Key Takeaways for Exam Success:**

- **Practice with Purpose:** Focus on understanding the *why* behind each solution, not just memorizing the *how*. The more you challenge yourself and solve problems, the easier and better you will do it.
- **Embrace Your Mistakes:** Every mistake is an opportunity to learn. Analyze what worked and what you can improve next time.
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