International Baccalaureate (IB) Diploma Programme Mathematics Analysis and Approaches Higher Level

Functions

The IB 7-Scorer's Ultimate Guide

Crafted Exclusively for High-Achieving IB Mathematics Students: April 2025 Edition

Mathematics Elevate Academy

Excellence in Advanced Mathematics Education

Rishabh Kumar

Founder, Mathematics Elevate Academy

Elite Mentor for IB Mathematics

Alumnus of Indian Institute of Technology Guwahati & Indian Statistical Institute

Visit me, Access Free Resources & Apply For Exclusive Personalized Mentorship

www.mathematicselevateacademy.com

www.linkedin.com/in/rishabh-kumar-iitg-isi/

Disclaimer

This document is provided for free for personal and educational use only. Commercial use, redistribution, or modification of this document is strictly prohibited. For permissions, you can contact mathematicselevateacademy001@gmail.com.

Introduction

The IB 7-Scorer's Ultimate Guide — April 2025 Edition is thoughtfully designed for IB DP Mathematics students striving to excel in IB DP Mathematics AA SL/HL, with a special emphasis on Functions and conceptual mastery. This guide offers a wealth of expertly crafted high-level Functions problems, conceptual challenges, and much more.

Explore examiner-style solutions, detailed marking scheme breakdowns, and insightful commentary on common errors to refine your problem-solving skills. Each problem is designed to test your grasp of Functions concepts.

This guide goes beyond the IB syllabus, offering enrichment problems that challenge your mathematical thinking and prepare you for Olympiads and university-level mathematics. The solutions are presented with step-by-step clarity, expert insights, and advanced techniques, ensuring a comprehensive and engaging learning experience.

For answers or detailed solutions, keep following me — they will be available soon! For personalized learning, book a one-on-one mentorship session with me to receive customized guidance on mastering IB DP Mathematics AA/AI HL, Functions, or even Olympiad-level problems. Together, we will build the confidence and skills you need to excel.

Check Your Understanding!

1 Concept of a function

Problem 1.1: Function Notation

Problem Statement

1. If $f(x) = 2x + 3$, evaluate:

1. $f(2)$
2. $f(-1)$
3. $f(a + 1)$

2. Given $g(x) = x^2 - 4x + 5$, find:

1. $g(0)$
2. $g(3)$
3. $g(t - 2)$

3. If $h(x) = \frac{1}{x}$, find $h(2)$ and $h(-3)$.

—

Problem 1.2: Domain of a Function

Problem Statement

1. Find the domain of the following functions:

1. $f(x) = \frac{1}{x}$
2. $g(x) = \sqrt{x - 2}$
3. $h(x) = \frac{\sqrt{x+3}}{x-1}$

2. Determine the domain of $f(x) = \ln(x - 1)$.

3. Explain why the domain of $f(x) = \sqrt{x}$ is $x \geq 0$.

—

Problem 1.3: Range of a Function

Problem Statement

1. Find the range of the following functions:

1. $f(x) = x^2$
2. $g(x) = \sqrt{x-1}$
3. $h(x) = \frac{1}{x}$

2. Determine the range of $f(x) = 2x + 3$ for $x \in [0, 5]$.

3. Explain why the range of $f(x) = \ln(x)$ is $(-\infty, \infty)$.

—

Problem 1.4: Inverse Functions

Problem Statement

1. Find the inverse of the following functions:

1. $f(x) = 2x + 3$
2. $g(x) = \frac{1}{x}$
3. $h(x) = x^2$, for $x \geq 0$

2. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ for $f(x) = 2x + 3$.

3. Explain why the function $f(x) = x^2$ does not have an inverse unless its domain is restricted.

—

Problem 1.5: Graph of an Inverse Function

Problem Statement

1. Sketch the graph of $f(x) = 2x + 3$ and its inverse $f^{-1}(x)$. Show that the graphs are reflections of each other in the line $y = x$.

2. Sketch the graph of $f(x) = x^2$ (for $x \geq 0$) and its inverse $f^{-1}(x) = \sqrt{x}$.

3. Explain why the graph of an inverse function is always a reflection of the original function in the line $y = x$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Function Notation**: If $f(x)$ is a function, $f(a)$ represents the value of the function when $x = a$.
2. **Domain**: The set of all possible input values (x) for which the function is defined.
3. **Range**: The set of all possible output values $(f(x))$ of the function.
4. **Inverse Function**: The inverse of a function $f(x)$, denoted $f^{-1}(x)$, satisfies:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

5. **Graph of an Inverse Function**: The graph of $f^{-1}(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$.

—

Marking Guidelines

Marking Scheme

Problem 1: Function Notation

- Correct evaluation of function values [2 marks per part].
- Accurate substitution and simplification [1 mark per part].

Problem 2: Domain of a Function

- Correct identification of restrictions on x [2 marks per part].
- Valid explanation of domain restrictions [2 marks].

Problem 3: Range of a Function

- Correct calculation of range [2 marks per part].
- Valid explanation of range properties [2 marks].

Problem 4: Inverse Functions

- Correct calculation of inverse functions [3 marks per part].
- Verification of inverse properties [2 marks].
- Valid explanation of domain restrictions for inverses [2 marks].

Problem 5: Graph of an Inverse Function

- Accurate sketch of the function and its inverse [3 marks per part].
- Correct identification of reflection in $y = x$ [2 marks].
- Valid explanation of reflection property [2 marks].

2 Graph of a Function

Problem 2.1: Creating a Sketch from Given Information

Problem Statement

1. Sketch the graph of a quadratic function $f(x) = ax^2 + bx + c$ given the following features:

- Vertex at $(1, -2)$
- Passes through the point $(0, 1)$
- Opens upwards

2. Sketch the graph of a cubic function $g(x)$ with the following features:

- Roots at $x = -2$, $x = 0$, and $x = 3$
- Passes through the point $(1, -4)$
- Positive leading coefficient

3. Sketch the graph of an exponential function $h(x) = a \cdot b^x$ given:

- Passes through the points $(0, 2)$ and $(1, 6)$
- Horizontal asymptote at $y = 0$

—

Problem 2.2: Using Technology to Graph Functions

Problem Statement

1. Use a graphing calculator or software to sketch the graph of the following functions:

1. $f(x) = x^3 - 3x^2 + 2x$
2. $g(x) = \frac{1}{x}$
3. $h(x) = \sin(x)$ for $x \in [0, 2\pi]$

2. Identify the key features of the graph of $f(x) = x^3 - 3x^2 + 2x$:

- Roots
- Turning points
- End behavior

3. Use technology to graph $f(x) = e^x$ and $g(x) = \ln(x)$ on the same set of axes. Identify their points of intersection and describe their relationship.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Quadratic Function**:

$$f(x) = ax^2 + bx + c$$

Key features include the vertex, axis of symmetry, and roots.

2. **Cubic Function**:

$$f(x) = ax^3 + bx^2 + cx + d$$

Key features include roots, turning points, and end behavior.

3. **Exponential Function**:

$$f(x) = a \cdot b^x$$

Key features include the horizontal asymptote and growth/decay behavior.

4. **Using Technology**: Graphing calculators or software can be used to plot functions, identify key features, and analyze their behavior.

—

Marking Guidelines

Marking Scheme

Problem 2.1: Creating a Sketch from Given Information

- Correct identification of key features (e.g., vertex, roots, asymptotes) [2 marks per part].
- Accurate sketch of the graph based on the given features [3 marks per part].

Problem 2.2: Using Technology to Graph Functions

- Correct use of technology to plot the graph [2 marks per part].
- Accurate identification of key features (e.g., roots, turning points, asymptotes) [2 marks per part].
- Valid interpretation of the relationship between functions [2 marks].

3 Key Features of Graphs

Problem 3.1: Finding Vertices and Lines of Symmetry

Problem Statement

1. Use your GDC to find the vertex and line of symmetry of the following quadratic functions:

1. $f(x) = x^2 - 4x + 3$
2. $g(x) = -2x^2 + 8x - 5$
3. $h(x) = 3x^2 - 6x + 1$

2. For the function $f(x) = x^3 - 3x^2 + 2x$, use your GDC to:

- Find the local maximum and minimum values.
- Identify the intervals where the function is increasing and decreasing.

—

Problem 3.2: Finding Vertical and Horizontal Asymptotes

Problem Statement

1. Use your GDC to find the vertical and horizontal asymptotes of the following functions:

1. $f(x) = \frac{1}{x}$
2. $g(x) = \frac{2x+1}{x-3}$
3. $h(x) = \frac{x^2-4}{x^2-1}$

2. Sketch the graph of $f(x) = \frac{1}{x-2}$ and identify its vertical and horizontal asymptotes.

3. Explain why the function $f(x) = e^x$ has a horizontal asymptote but no vertical asymptote.

—

Problem 3.3: Finding Zeros of Functions or Roots of Equations

Problem Statement

1. Use your GDC to find the zeros of the following functions:

1. $f(x) = x^2 - 5x + 6$
2. $g(x) = x^3 - 4x^2 + x + 6$
3. $h(x) = \sin(x)$ for $x \in [0, 2\pi]$

2. Solve the equation $x^3 - 2x^2 - x + 2 = 0$ using your GDC.

3. Verify that the roots of $f(x) = x^2 - 4$ are $x = \pm 2$ by graphing the function.

—

Problem 3.4: Finding Points of Intersection of Two Curves or Lines

Problem Statement

1. Use your GDC to find the points of intersection of the following pairs of functions:

1. $f(x) = x^2$ and $g(x) = 2x + 3$
2. $f(x) = \sin(x)$ and $g(x) = \cos(x)$ for $x \in [0, 2\pi]$
3. $f(x) = e^x$ and $g(x) = 2x + 1$

2. Find the intersection points of the lines $y = 3x - 2$ and $y = -x + 4$ using your GDC.

3. Explain why two parallel lines do not have any points of intersection.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Vertex of a Quadratic Function**: For $f(x) = ax^2 + bx + c$, the vertex is at:
$$x = -\frac{b}{2a}$$

2. **Asymptotes**:
   - Vertical asymptotes occur where the denominator of a rational function is zero.
   - Horizontal asymptotes are determined by the behavior of the function as $x \to \pm\infty$.

3. **Zeros of a Function**: The zeros of a function $f(x)$ are the values of x for which $f(x) = 0$.

4. **Intersection of Two Curves**: The points of intersection of $f(x)$ and $g(x)$ are the solutions to the equation:
$$f(x) = g(x)$$

—

Marking Guidelines

Marking Scheme

Problem 3.1: Finding Vertices and Lines of Symmetry

- Correct identification of vertex and line of symmetry [2 marks per part].
- Accurate use of GDC to find local maxima and minima [3 marks per part].

Problem 3.2: Finding Vertical and Horizontal Asymptotes

- Correct identification of vertical and horizontal asymptotes [2 marks per part].
- Valid explanation of asymptotic behavior [2 marks].

Problem 3.3: Finding Zeros of Functions or Roots of Equations

- Correct identification of zeros using GDC [2 marks per part].
- Accurate verification of roots [2 marks].

Problem 3.4: Finding Points of Intersection of Two Curves or Lines

- Correct identification of intersection points using GDC [2 marks per part].
- Valid explanation of intersection properties [2 marks].

4 Composite Functions

Problem 4.1: Finding the Composite Function

Problem Statement

1. Given the functions $f(x) = 2x + 3$ and $g(x) = x^2$, find:

1. $(f \circ g)(x)$
2. $(g \circ f)(x)$

2. If $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find:

1. $(f \circ g)(x)$
2. $(g \circ f)(x)$

3. For $f(x) = \frac{1}{x}$ and $g(x) = x + 2$, find $(f \circ g)(x)$ and simplify.

—

Problem 4.2: Domain of a Composite Function

Problem Statement

1. Find the domain of the composite function $(f \circ g)(x)$ for the following:

1. $f(x) = \sqrt{x}$ and $g(x) = x - 2$
2. $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 4$
3. $f(x) = \ln(x)$ and $g(x) = e^x$

2. Explain why the domain of $(f \circ g)(x)$ is restricted by the domains of both $f(x)$ and $g(x)$.

3. Determine the domain of $(g \circ f)(x)$ for $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Composite Function**: The composite function $(f \circ g)(x)$ is defined as:
$$(f \circ g)(x) = f(g(x))$$
2. **Domain of a Composite Function**: The domain of $(f \circ g)(x)$ is the set of all x such that:
   - x is in the domain of $g(x)$, and
   - $g(x)$ is in the domain of $f(x)$.

—

Marking Guidelines

Marking Scheme

Problem 4.1: Finding the Composite Function

- Correct substitution of $g(x)$ into $f(x)$ or vice versa [2 marks per part].
- Accurate simplification of the composite function [2 marks per part].

Problem 4.2: Domain of a Composite Function

- Correct identification of domain restrictions for $g(x)$ and $f(g(x))$ [2 marks per part].
- Valid explanation of domain restrictions [2 marks].

5 Inverse Functions

Problem 5.1: Finding the Inverse of a Function

Problem Statement

1. Find the inverse of the following functions:

1. $f(x) = 2x + 3$
2. $g(x) = \frac{1}{x}$
3. $h(x) = x^2$, for $x \geq 0$

2. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ for $f(x) = 3x - 5$.

3. For $f(x) = \sqrt{x - 1}$, find $f^{-1}(x)$ and state its domain and range.

—

Problem 5.2: One-to-One Functions and Inverses

Problem Statement

1. Determine whether the following functions are one-to-one:

1. $f(x) = x^2$
2. $g(x) = 2x + 1$
3. $h(x) = |x|$

2. Explain why a function must be one-to-one to have an inverse.

3. Restrict the domain of $f(x) = x^2$ so that it has an inverse, and find the inverse function.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Inverse Function**: The inverse of a function $f(x)$, denoted $f^{-1}(x)$, satisfies:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

2. **One-to-One Function**: A function is one-to-one if each y-value in the range corresponds to exactly one x-value in the domain.

3. **Steps to Find the Inverse**:

   (a) Replace $f(x)$ with y: $y = f(x)$.

   (b) Swap x and y: $x = f(y)$.

   (c) Solve for y in terms of x.

   (d) Replace y with $f^{-1}(x)$.

—

Marking Guidelines

Marking Scheme

Problem 5.1: Finding the Inverse of a Function

- Correct application of steps to find the inverse [3 marks per part].
- Verification of inverse properties [2 marks per part].
- Accurate identification of domain and range [2 marks].

Problem 5.2: One-to-One Functions and Inverses

- Correct determination of whether a function is one-to-one [2 marks per part].
- Valid explanation of why a function must be one-to-one to have an inverse [3 marks].
- Accurate restriction of domain and calculation of inverse [3 marks].

6 Quadratic Functions

Problem 6.1: Graph and Y-intercept

Problem Statement

1. For each quadratic function, identify the shape (opens upward/downward) and y-intercept:

1. $f(x) = 2x^2 - 3x + 1$
2. $g(x) = -x^2 + 4x - 5$
3. $h(x) = \frac{1}{2}x^2 + 2x - 3$

2. Find the axis of symmetry for each quadratic function:

1. $f(x) = x^2 + 6x + 2$
2. $g(x) = -2x^2 + 8x - 7$

3. Explain why the graph of $f(x) = ax^2 + bx + c$ opens upward when $a > 0$ and downward when $a < 0$.

—

Problem 6.2: X-intercepts through Factoring

Problem Statement

1. Find the x-intercepts by factoring:

1. $f(x) = x^2 - 5x + 6$
2. $g(x) = 2x^2 - 3x - 5$
3. $h(x) = x^2 - 4$

2. For $f(x) = x^2 + kx + 4$, find the value of k for which the function has exactly one x-intercept.

3. Explain why some quadratic functions have no real x-intercepts.

—

Problem 6.3: Vertex Form and Line of Symmetry

Problem Statement

1. Complete the square to find the vertex and line of symmetry:

1. $f(x) = x^2 + 6x + 5$
2. $g(x) = 2x^2 - 8x + 7$
3. $h(x) = -x^2 + 4x - 1$

2. Convert $f(x) = x^2 - 4x + 1$ to the form $a(x - h)^2 + k$ and identify:

- The vertex (h, k)
- The axis of symmetry
- Whether the function has a maximum or minimum value

3. Given the vertex form $f(x) = 2(x - 3)^2 - 4$, find:

- The vertex
- The axis of symmetry
- The y-intercept

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Standard Form**:

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$

2. **Axis of Symmetry**:

$$x = -\frac{b}{2a}$$

3. **Factored Form**:

$$f(x) = a(x-p)(x-q)$$

where p and q are the x-intercepts

4. **Vertex Form**:

$$f(x) = a(x-h)^2 + k$$

where (h,k) is the vertex

5. **Completing the Square**:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$$

—

Marking Guidelines

Marking Scheme

Problem 6.1: Graph and Y-intercept

- Correct identification of shape [1 mark per part]
- Accurate determination of y-intercept [1 mark per part]
- Correct calculation of axis of symmetry [2 marks per part]
- Valid explanation of opening direction [2 marks]

Problem 6.2: X-intercepts through Factoring

- Correct factorization [2 marks per part]
- Accurate identification of x-intercepts [1 mark per part]
- Valid solution for special case [3 marks]
- Clear explanation of no real roots case [2 marks]

Problem 6.3: Vertex Form and Line of Symmetry

- Correct completion of square [2 marks per part]
- Accurate identification of vertex [1 mark per part]
- Correct identification of line of symmetry [1 mark per part]
- Valid determination of maximum/minimum [2 marks]

7 Quadratic Equations and Inequalities

Problem 7.1: Solving Quadratic Equations by Factoring

Problem Statement

1. Solve the following quadratic equations by factoring:

1. $x^2 - 7x + 12 = 0$
2. $2x^2 + 5x - 3 = 0$
3. $x^2 - 4 = 0$

2. Solve $(x - 2)(x + 3) = 6$.

3. Find the value of k for which $x^2 + kx + 4 = 0$ has equal roots.

—

Problem 7.2: Solving by Completing the Square

Problem Statement

1. Solve the following equations by completing the square:

1. $x^2 + 6x + 5 = 0$
2. $2x^2 - 8x + 7 = 0$
3. $x^2 - 4x = 5$

2. Use completing the square to show that $x^2 + 2x + 5 = 0$ has no real solutions.

3. Find the minimum value of $f(x) = x^2 + 4x + 7$ by completing the square.

—

Problem 7.3: Using the Quadratic Formula

Problem Statement

1. Solve using the quadratic formula:

1. $2x^2 - 7x + 3 = 0$
2. $3x^2 + 2x + 1 = 0$
3. $x^2 + x + 1 = 0$

2. For the equation $ax^2 + bx + c = 0$:

- Explain what $b^2 - 4ac$ tells us about the nature of roots
- Find conditions on a, b, and c for real and equal roots

—

Problem 7.4: Solving Quadratic Inequalities

Problem Statement

1. Solve the following quadratic inequalities:

1. $x^2 - 5x + 6 > 0$
2. $2x^2 + 3x - 2 \leq 0$
3. $(x-1)(x+2) < 0$

2. Solve $\frac{x^2-4}{x+2} \geq 0$, stating any restrictions on x.

3. Find the values of x for which $f(x) = x^2 - 4x + 3$ is positive.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Quadratic Formula**: For $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. **Discriminant**: $b^2 - 4ac$ determines the nature of roots:
   - > 0: Two distinct real roots
   - $= 0$: One repeated real root
   - < 0: No real roots

3. **Completing the Square**:

$$x^2 + bx = (x + \frac{b}{2})^2 - (\frac{b}{2})^2$$

4. **Quadratic Inequalities**: Sign analysis using:
   - Critical points (roots)
   - Test points in intervals
   - Sign diagram

—

Marking Guidelines

Marking Scheme

Problem 7.1: Solving by Factoring

- Correct factorization [2 marks per part]
- Accurate solution [1 mark per part]
- Valid determination of special cases [2 marks]

Problem 7.2: Completing the Square

- Correct completion of square process [2 marks per part]
- Accurate solution [2 marks per part]
- Valid explanation of no real solutions [2 marks]

Problem 7.3: Quadratic Formula

- Correct application of formula [2 marks per part]
- Accurate interpretation of discriminant [2 marks]
- Valid analysis of conditions [2 marks]

Problem 7.4: Quadratic Inequalities

- Correct identification of critical points [2 marks per part]
- Accurate interval analysis [2 marks per part]
- Valid solution with restrictions [2 marks]

8 Quadratic Discriminant

Problem 8.1: Nature of Roots Using the Discriminant

Problem Statement

1. For each quadratic equation, calculate the discriminant $\Delta = b^2 - 4ac$ and determine the nature of the roots:

1. $x^2 - 5x + 6 = 0$
2. $2x^2 + 4x + 2 = 0$
3. $x^2 + 2x + 5 = 0$

2. Explain why the discriminant $\Delta > 0$ implies two distinct real roots, $\Delta = 0$ implies one real root, and $\Delta < 0$ implies no real roots.

3. Verify the nature of the roots of $f(x) = x^2 - 4x + 4$ using the discriminant and solve the equation to confirm your result.

—

Problem 8.2: Finding Parameter Values Using the Discriminant

Problem Statement

1. Find the set of values of k for which the quadratic equation $x^2 + kx + 4 = 0$ has:

1. Two distinct real roots
2. One real root
3. No real roots

2. For the quadratic equation $2x^2 + px + 3 = 0$, find the range of values of p such that the equation has:

1. Two distinct real roots
2. No real roots

3. Determine the value of k such that the quadratic equation $kx^2 - 4x + 1 = 0$ has exactly one real root.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Discriminant**: For a quadratic equation $ax^2 + bx + c = 0$, the discriminant is given by:

$$\Delta = b^2 - 4ac$$

2. **Nature of Roots**:

   - $\Delta > 0$: Two distinct real roots
   - $\Delta = 0$: One real root (repeated root)
   - $\Delta < 0$: No real roots (complex roots)

3. **Using the Discriminant to Find Parameter Values**: To find the range of a parameter (e.g., k or p), solve the inequality:

$$b^2 - 4ac > 0, \quad b^2 - 4ac = 0, \quad \text{or} \quad b^2 - 4ac < 0$$

depending on the required nature of the roots.

—

Marking Guidelines

Marking Scheme

Problem 8.1: Nature of Roots Using the Discriminant

- Correct calculation of the discriminant [2 marks per part]
- Accurate determination of the nature of roots [1 mark per part]
- Valid explanation of the relationship between Δ and the roots [2 marks]

Problem 8.2: Finding Parameter Values Using the Discriminant

- Correct setup of the discriminant inequality [2 marks per part]
- Accurate solution of the inequality [2 marks per part]
- Valid interpretation of the parameter range [2 marks per part]

9 Rational Functions

Problem 9.1: The Reciprocal Function

Problem Statement

1. For the reciprocal function $f(x) = \frac{1}{x}$:

1. Sketch the graph
2. Find the equations of horizontal and vertical asymptotes
3. Identify the domain and range
4. Prove that $f(f(x)) = x$, showing it is self-inverse

2. For $g(x) = \frac{k}{x}$, where $k > 0$:

1. Describe how the graph differs from $f(x) = \frac{1}{x}$
2. Find its asymptotes
3. Determine if it is self-inverse

3. Find the points where the graph of $f(x) = \frac{1}{x}$ intersects:

1. The line $y = x$
2. The line $y = -x$

Problem 9.2: Rational Functions of the Form $f(x) = \frac{ax+b}{cx+d}$

Problem Statement

1. For each rational function:

- Find the vertical and horizontal asymptotes
- Find any x and y intercepts
- Sketch the graph

1. $f(x) = \frac{2x+3}{x-1}$
2. $g(x) = \frac{3x-6}{2x+4}$
3. $h(x) = \frac{x+2}{x-2}$

2. For $f(x) = \frac{ax+b}{cx+d}$:

1. Find the general form of the horizontal asymptote
2. Under what conditions will there be no horizontal asymptote?
3. When will the horizontal asymptote be $y = 1$?

3. Find the values of k for which the graph of $f(x) = \frac{kx+2}{x-1}$ passes through the point $(2, 3)$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Reciprocal Function**:

$$f(x) = \frac{1}{x}, \quad x \neq 0$$

Properties:

- Vertical asymptote: $x = 0$
- Horizontal asymptote: $y = 0$
- Domain: $x \in \mathbb{R}, x \neq 0$
- Range: $y \in \mathbb{R}, y \neq 0$

2. **General Rational Function**:

$$f(x) = \frac{ax + b}{cx + d}, \quad cx + d \neq 0$$

Properties:

- Vertical asymptote: $x = -\frac{d}{c}$
- Horizontal asymptote: $y = \frac{a}{c}$ (when $c \neq 0$)
- x-intercept: $x = -\frac{b}{a}$ (when $a \neq 0$)
- y-intercept: $y = \frac{b}{d}$ (when $d \neq 0$)

—

Marking Guidelines

Marking Scheme

Problem 9.1: The Reciprocal Function

- Accurate graph sketch showing both branches [2 marks]
- Correct identification of asymptotes [2 marks]
- Valid proof of self-inverse property [3 marks]
- Correct domain and range [2 marks]
- Accurate intersection points [2 marks per point]

Problem 9.2: General Rational Functions

- Correct identification of asymptotes [2 marks per function]
- Accurate determination of intercepts [2 marks per function]
- Complete and accurate graph sketches [3 marks per function]
- Valid analysis of horizontal asymptote conditions [2 marks]
- Correct solution for parameter value [3 marks]

Additional Points

- Clear labeling of all key features on graphs [1 mark per graph]
- Correct use of mathematical notation [1 mark]
- Logical presentation of solutions [1 mark]

10 Exponential and Logarithmic Functions

Problem 10.1: Exponential Functions and Their Graphs

Problem Statement

1. Sketch the graphs of the following exponential functions:

1. $f(x) = 2^x$
2. $g(x) = (\frac{1}{2})^x$
3. $h(x) = e^x$

For each function, identify:

- y-intercept
- Horizontal asymptote
- Domain and range
- Intervals of increase/decrease

2. Compare and contrast the graphs of $f(x) = 2^x$ and $g(x) = 2^{-x}$.

3. Find the value of k such that the graph of $f(x) = e^x$ passes through the point $(2, k)$.

—

Problem 10.2: Logarithmic Functions and Their Graphs

Problem Statement

1. Sketch the graphs of the following logarithmic functions:

1. $f(x) = \log_2(x)$
2. $g(x) = \ln(x)$
3. $h(x) = \log_{\frac{1}{2}}(x)$

For each function, identify:

- x-intercept
- Vertical asymptote
- Domain and range
- Intervals of increase/decrease

2. Explain why all logarithmic functions have a vertical asymptote at $x = 0$.

3. Find the coordinates of the point where $f(x) = \ln(x)$ intersects the line $y = x$.

—

Problem 10.3: Relationship Between Exponential and Logarithmic Functions

Problem Statement

1. Show that the graphs of $y = 2^x$ and $y = \log_2(x)$ are reflections of each other in the line $y = x$.

2. Prove that $\log_a(x)$ and a^x are inverse functions.

3. Given the point $(3, 8)$ lies on the graph of $y = 2^x$, find a point that must lie on the graph of $y = \log_2(x)$.

—

Problem 10.4: Converting Between Bases

Problem Statement

1. Express the following in terms of e:

1. 2^x
2. 3^x
3. 10^x

2. Solve the equation $2^x = 8$ using natural logarithms.

3. Express $\log_2(x)$ in terms of natural logarithms.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Exponential Functions**:

$$f(x) = a^x, \quad a > 0, a \neq 1$$

Properties:

- Domain: $\mathbb{R}$
- Range: $(0, \infty)$
- y-intercept: 1
- Horizontal asymptote: $y = 0$ (if $0 < a < 1$)

2. **Logarithmic Functions**:

$$f(x) = \log_a(x), \quad a > 0, a \neq 1$$

Properties:

- Domain: $(0, \infty)$
- Range: $\mathbb{R}$
- x-intercept: 1
- Vertical asymptote: $x = 0$

3. **Base Conversion**:

$$a^x = e^{x \ln(a)}$$

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

—

Marking Guidelines

Marking Scheme

Problem 10.1: Exponential Functions

- Accurate graph sketches [2 marks per graph]
- Correct identification of key features [1 mark per feature]
- Valid comparison of related functions [2 marks]

Problem 10.2: Logarithmic Functions

- Accurate graph sketches [2 marks per graph]
- Correct identification of key features [1 mark per feature]
- Valid explanation of asymptotic behavior [2 marks]

Problem 10.3: Relationship Between Functions

- Valid proof of reflection property [3 marks]
- Correct proof of inverse relationship [3 marks]
- Accurate determination of corresponding points [2 marks]

Problem 10.4: Base Conversion

- Correct conversion to base e [2 marks per expression]
- Accurate solution using natural logarithms [3 marks]
- Valid expression in terms of natural logarithms [2 marks]

11 Solving Equations Analytically

Problem 11.1: Solving Equations by Factorizing

Problem Statement

1. Solve the following equations by factorizing:

1. $x^2 - 7x + 12 = 0$
2. $2x^2 + 5x - 3 = 0$
3. $x^2 - 4 = 0$

2. Solve $(x - 3)(x + 2) = 0$.

3. Solve $x^3 - 3x^2 - 4x + 12 = 0$ by factorizing.

—

Problem 11.2: Solving Disguised Quadratics

Problem Statement

1. Solve the following disguised quadratic equations:

1. $x^4 - 5x^2 + 4 = 0$
2. $2y^2 - 3y - 2 = 0$, where $y = x^2$
3. $e^{2x} - 5e^x + 6 = 0$

2. Solve $x^{2/3} - 5x^{1/3} + 6 = 0$.

3. Solve $\sin^2(x) - \sin(x) - 2 = 0$ for $x \in [0, 2\pi]$.

—

Problem 11.3: Solving Equations That Lead to Quadratics

Problem Statement

1. Solve the following equations that lead to quadratics:

1. $\frac{1}{x} + \frac{2}{x+1} = 1$
2. $\sqrt{x+2} + x = 4$
3. $\ln(x) + \ln(x-1) = \ln(6)$

2. Solve $x + \frac{1}{x} = 5$.

3. Solve $2^{x+1} + 2^x = 12$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Factorizing Quadratics**: For $ax^2 + bx + c = 0$, factorize into:

$$(px+q)(rx+s) = 0$$

2. **Disguised Quadratics**: Rewrite higher-degree or transformed equations into standard quadratic form:

$$u^2 + bu + c = 0, \quad \text{where } u = x^n, e^x, \sin(x), \text{etc.}$$

3. **Equations Leading to Quadratics**: Rearrange equations to form a quadratic by:
   - Clearing fractions
   - Squaring both sides (if necessary)
   - Using logarithmic or exponential properties

—

Marking Guidelines

Marking Scheme

Problem 11.1: Solving by Factorizing

- Correct factorization [2 marks per part]
- Accurate solution of the equation [1 mark per part]

Problem 11.2: Solving Disguised Quadratics

- Correct substitution to rewrite the equation as a quadratic [2 marks per part]
- Accurate solution of the quadratic equation [2 marks per part]
- Valid back-substitution and verification of solutions [2 marks per part]

Problem 11.3: Solving Equations That Lead to Quadratics

- Correct rearrangement to form a quadratic equation [2 marks per part]
- Accurate solution of the quadratic equation [2 marks per part]
- Valid interpretation of solutions in the context of the original equation [2 marks per part]

12 Solving Equations Graphically

Problem 12.1: Solving Equations of the Form $f(x) = 0$

Problem Statement

1. Use the graphing feature on your GDC to solve the following equations:

1. $x^2 - 5x + 6 = 0$
2. $e^x - 3 = 0$
3. $\sin(x) - 0.5 = 0$ for $x \in [0, 2\pi]$

2. Sketch the graph of $f(x) = x^3 - 4x + 1$ and use your GDC to find the roots of the equation $f(x) = 0$.

3. Explain how the graphing feature of your GDC can be used to approximate solutions to equations that cannot be solved analytically, such as $x^5 - x + 1 = 0$.

—

Problem 12.2: Solving Equations of the Form $f(x) = g(x)$

Problem Statement

1. Use the graphing feature on your GDC to solve the following equations by finding the points of intersection of the graphs of $f(x)$ and $g(x)$:

1. $x^2 = 2x + 3$
2. $e^x = 2x + 1$
3. $\sin(x) = \cos(x)$ for $x \in [0, 2\pi]$

2. Sketch the graphs of $f(x) = x^2$ and $g(x) = 3x - 4$, and use your GDC to find their points of intersection.

3. Explain why the solutions to $f(x) = g(x)$ correspond to the x-coordinates of the points of intersection of the graphs of $f(x)$ and $g(x)$.

—

Key Concepts and Steps for Using GDC

Key Concepts and Steps

1. **Solving $f(x) = 0$**:
   - Enter the function $f(x)$ into your GDC.
   - Use the "zero" or "root" feature to find the x-values where $f(x) = 0$.
   - Verify the solutions by checking the graph.
2. **Solving $f(x) = g(x)$**:
   - Enter both $f(x)$ and $g(x)$ into your GDC.
   - Use the "intersection" feature to find the x-coordinates of the points where the graphs intersect.
   - Verify the solutions by checking the graph.
3. **Graphing Tips**:
   - Adjust the viewing window to ensure all roots or intersections are visible.
   - Use zoom features to refine the accuracy of your solutions.

—

Marking Guidelines

Marking Scheme

Problem 12.1: Solving $f(x) = 0$

- Correct use of GDC to find roots [2 marks per equation]
- Accurate sketch of the graph [2 marks per graph]
- Valid explanation of the GDC process [2 marks]

Problem 12.2: Solving $f(x) = g(x)$

- Correct use of GDC to find points of intersection [2 marks per equation]
- Accurate sketch of the graphs [2 marks per graph]
- Valid explanation of the relationship between intersections and solutions [2 marks]

Additional Points

- Clear labeling of graphs and key points [1 mark per graph]
- Logical presentation of solutions [1 mark]

13 Transformations of Graphs

Problem 13.1: Translations

Problem Statement

1. For the function $f(x) = x^2$, sketch the graphs of:

1. $y = f(x) + 3$
2. $y = f(x) - 2$
3. $y = f(x - 4)$
4. $y = f(x + 1)$

For each transformation, describe the type of translation and the direction.

2. The graph of $g(x)$ is obtained by translating $f(x) = \sqrt{x}$ vertically by 5 units and horizontally by -3 units. Write the equation of $g(x)$ and sketch its graph.

3. Explain the difference between $y = f(x) + b$ and $y = f(x - a)$ in terms of graph transformations.

—

Problem 13.2: Stretches

Problem Statement

1. For the function $f(x) = x^2$, sketch the graphs of:

1. $y = 2f(x)$
2. $y = \frac{1}{2}f(x)$
3. $y = f(2x)$
4. $y = f\left(\frac{1}{2}x\right)$

For each transformation, describe the type of stretch and the scale factor.

2. The graph of $g(x)$ is obtained by applying a vertical stretch with scale factor 3 and a horizontal stretch with scale factor $\frac{1}{2}$ to $f(x) = \sin(x)$. Write the equation of $g(x)$ and sketch its graph.

3. Explain why $y = pf(x)$ results in a vertical stretch, while $y = f(qx)$ results in a horizontal stretch with scale factor $\frac{1}{q}$.

—

Problem 13.3: Reflections

Problem Statement

1. For the function $f(x) = x^3$, sketch the graphs of:

1. $y = -f(x)$
2. $y = f(-x)$

For each transformation, describe the type of reflection and the axis of reflection.

2. The graph of $g(x)$ is obtained by reflecting $f(x) = e^x$ in both the x-axis and the y-axis. Write the equation of $g(x)$ and sketch its graph.

3. Explain how the transformations $y = -f(x)$ and $y = f(-x)$ affect the domain and range of $f(x)$.

—

Problem 13.4: Composite Transformations

Problem Statement

1. For the function $f(x) = x^2$, sketch the graphs of:

1. $y = 2f(x) + 3$
2. $y = f(2x) - 1$

For each transformation, describe the sequence of transformations applied to $f(x)$.

2. The graph of $g(x)$ is obtained by applying a horizontal translation of 4 units to the right and a vertical stretch with scale factor 2 to $f(x) = \ln(x)$. Write the equation of $g(x)$ and sketch its graph.

3. Explain how to apply one horizontal and one vertical transformation to a graph in sequence, and describe how the order of transformations affects the final graph.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Translations**:
   - $y = f(x) + b$: Vertical translation by b units (up if $b > 0$, down if $b < 0$)
   - $y = f(x - a)$: Horizontal translation by a units (right if $a > 0$, left if $a < 0$)
2. **Stretches**:
   - $y = pf(x)$: Vertical stretch with scale factor p (compression if $0 < p < 1$)
   - $y = f(qx)$: Horizontal stretch with scale factor $\frac{1}{q}$ (compression if $q > 1$)
3. **Reflections**:
   - $y = -f(x)$: Reflection in the x-axis
   - $y = f(-x)$: Reflection in the y-axis
4. **Composite Transformations**: Apply transformations in sequence, starting with horizontal transformations, followed by vertical transformations.

—

Marking Guidelines

Marking Scheme

Problem 13.1: Translations

- Correct identification of translation type and direction [2 marks per part]
- Accurate graph sketches [2 marks per graph]

Problem 13.2: Stretches

- Correct identification of stretch type and scale factor [2 marks per part]
- Accurate graph sketches [2 marks per graph]

Problem 13.3: Reflections

- Correct identification of reflection type and axis [2 marks per part]
- Accurate graph sketches [2 marks per graph]

Problem 13.4: Composite Transformations

- Correct description of transformation sequence [2 marks per part]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of the effect of transformation order [2 marks]

14 Graphs and Equations of Polynomials

Problem 14.1: Recognizing the Shapes of Polynomial Graphs

Problem Statement

1. Sketch the general shapes of the following polynomial functions:

1. $f(x) = x^2$
2. $g(x) = x^3$
3. $h(x) = x^4$

For each graph, identify:

- The degree of the polynomial
- The end behavior of the graph
- The number of turning points

2. Compare the graphs of $f(x) = x^3$ and $g(x) = -x^3$. Describe how the negative sign affects the graph.

3. Explain how the degree and leading coefficient of a polynomial determine the end behavior of its graph.

—

Problem 14.2: Zeros, Roots, and Factors

Problem Statement

1. Use factorization to find the zeros of the following polynomials and sketch their graphs:

1. $f(x) = x^3 - 3x^2 - 4x + 12$
2. $g(x) = x^4 - 5x^2 + 4$
3. $h(x) = x^3 + 2x^2 - x - 2$

2. For $f(x) = (x-1)^2(x+2)$:

- Find the zeros and their multiplicities
- Describe how the multiplicities affect the graph at each zero
- Sketch the graph of $f(x)$

3. Explain why a polynomial of degree n can have at most n real roots.

—

Problem 14.3: Finding the Equation of a Polynomial from Its Graph

Problem Statement

1. The graph of a cubic polynomial passes through the points $(0,0)$, $(1,2)$, and $(-1,-2)$. The graph has a turning point at $(0,0)$. Find the equation of the polynomial.

2. A quartic polynomial has zeros at $x = -2$, $x = 0$, and $x = 3$ (with $x = 3$ being a double root). The graph passes through the point $(1,-4)$. Find the equation of the polynomial.

3. Explain how the degree, zeros, and a given point on the graph can be used to determine the equation of a polynomial.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **General Form of a Polynomial**:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is the degree of the polynomial.

2. **End Behavior**:
   - If the degree n is even and the leading coefficient $a_n > 0$, both ends of the graph point upwards.
   - If the degree n is even and $a_n < 0$, both ends point downwards.
   - If the degree n is odd and $a_n > 0$, the left end points down and the right end points up.
   - If the degree n is odd and $a_n < 0$, the left end points up and the right end points down.

3. **Zeros and Multiplicities**:
   - A zero of multiplicity 1 crosses the x-axis.
   - A zero of even multiplicity touches the x-axis but does not cross it.
   - A zero of odd multiplicity greater than 1 crosses the x-axis with a flattening effect.

4. **Finding the Equation from a Graph**: Use the general form:

$$f(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k}$$

where r_i are the zeros, m_i are their multiplicities, and a is a scaling factor determined by a point on the graph.

—

Marking Guidelines

Marking Scheme

Problem 14.1: Recognizing Shapes of Polynomial Graphs

- Correct identification of degree and end behavior [2 marks per part]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of the effect of the leading coefficient [2 marks]

Problem 14.2: Zeros, Roots, and Factors

- Correct factorization and identification of zeros [2 marks per part]
- Accurate description of multiplicities and their effects [2 marks per part]
- Accurate graph sketches [2 marks per graph]

Problem 14.3: Finding the Equation of a Polynomial

- Correct use of zeros and multiplicities to form the equation [3 marks per part]
- Accurate determination of the scaling factor [2 marks per part]
- Valid explanation of the process [2 marks]

15 The Factor and Remainder Theorems

Problem 15.1: Using the Remainder Theorem

Problem Statement

1. Use the remainder theorem to find the remainder when the following polynomials are divided by $x - 2$:

1. $f(x) = x^3 - 4x^2 + 5x - 2$
2. $g(x) = 2x^4 - 3x^3 + x - 7$
3. $h(x) = x^3 + 2x^2 - x + 4$

2. Verify your results by performing synthetic or long division.

3. Explain why the remainder theorem states that the remainder when $f(x)$ is divided by $x - c$ is $f(c)$.

—

Problem 15.2: Using the Factor Theorem

Problem Statement

1. Use the factor theorem to determine whether the following are factors of the given polynomials:

1. $x-3$ for $f(x)=x^3-6x^2+11x-6$
2. $x+2$ for $g(x)=2x^3+3x^2-2x-4$
3. $x-1$ for $h(x)=x^4-3x^3+2x^2-x+1$

2. If $x-2$ is a factor of $f(x)=x^3-5x^2+px-6$, find the value of p.

3. Explain how the factor theorem is a special case of the remainder theorem.

—

Problem 15.3: Factorizing Polynomials Using the Factor Theorem

Problem Statement

1. Factorize the following polynomials completely using the factor theorem:

1. $f(x)=x^3-6x^2+11x-6$
2. $g(x)=2x^3+3x^2-2x-4$
3. $h(x)=x^4-5x^2+4$

2. Solve the equation $x^3-4x^2+x+6=0$ by factorizing the polynomial.

3. Explain why the factor theorem is useful for factorizing higher-degree polynomials.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Remainder Theorem**: If a polynomial $f(x)$ is divided by $x - c$, the remainder is $f(c)$.
2. **Factor Theorem**: If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
3. **Steps for Factorizing Polynomials Using the Factor Theorem**:
   (a) Use the factor theorem to find a root c such that $f(c) = 0$.
   (b) Divide $f(x)$ by $x - c$ using synthetic or long division.
   (c) Repeat the process for the resulting quotient until the polynomial is fully factorized.

—

Marking Guidelines

Marking Scheme

Problem 15.1: Using the Remainder Theorem

- Correct substitution into $f(c)$ to find the remainder [2 marks per part]
- Accurate verification using synthetic or long division [2 marks per part]
- Valid explanation of the remainder theorem [2 marks]

Problem 15.2: Using the Factor Theorem

- Correct determination of whether a given expression is a factor [2 marks per part]
- Accurate calculation of unknown coefficients [2 marks per part]
- Valid explanation of the relationship between the factor and remainder theorems [2 marks]

Problem 15.3: Factorizing Polynomials Using the Factor Theorem

- Correct identification of factors using the factor theorem [2 marks per part]
- Accurate division of the polynomial [2 marks per part]
- Complete and correct factorization [2 marks per part]
- Valid explanation of the usefulness of the factor theorem [2 marks]

16 Sum and Product of Roots

Problem 16.1: Finding the Sum and Product of Roots

Problem Statement

1. For each of the following polynomials, find the sum and product of the roots using the formulas:

1. $f(x) = x^2 - 5x + 6$
2. $g(x) = 2x^2 + 3x - 4$
3. $h(x) = x^3 - 6x^2 + 11x - 6$

2. Verify your results for $f(x) = x^2 - 5x + 6$ by solving the equation and calculating the sum and product of the roots directly.

3. Explain why the sum of the roots of a quadratic equation $ax^2+bx+c = 0$ is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$.

—

Problem 16.2: Finding a Polynomial Whose Roots Are a Function of the Original Roots

Problem Statement

1. The roots of $f(x) = x^2 - 5x + 6$ are r_1 and r_2. Find a polynomial whose roots are:

1. $r_1 + 1$ and $r_2 + 1$
2. $2r_1$ and $2r_2$
3. $\frac{1}{r_1}$ and $\frac{1}{r_2}$

2. The roots of $g(x) = x^3 - 6x^2 + 11x - 6$ are r_1, r_2, and r_3. Find a polynomial whose roots are:

1. r_1^2, r_2^2, and r_3^2
2. $r_1 + 2$, $r_2 + 2$, and $r_3 + 2$

3. Explain the process of finding a new polynomial when the roots are transformed by a function of the original roots.

—

Problem 16.3: Finding a Polynomial with Given Roots

Problem Statement

1. Find a polynomial with the following roots:

1. 2 and -3
2. 1, -2, and 3
3. $\frac{1}{2}$, $-\frac{1}{3}$, and 4

2. Find a polynomial with roots r_1 and r_2 such that $r_1 + r_2 = 5$ and $r_1 r_2 = 6$.

3. Explain how the general form of a polynomial with roots $r_1, r_2, \ldots, r_n$ is constructed.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Sum and Product of Roots**: For a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$:
   - Sum of the roots:
     $$\text{Sum} = -\frac{a_{n-1}}{a_n}$$
   - Product of the roots:
     $$\text{Product} = (-1)^n \frac{a_0}{a_n}$$
2. **Finding a Polynomial with Transformed Roots**: If the roots of $f(x)$ are $r_1, r_2, \ldots, r_n$, and the new roots are $g(r_1), g(r_2), \ldots, g(r_n)$, substitute $x = g(r)$ into $f(x)$ and simplify.
3. **General Form of a Polynomial with Given Roots**: If the roots are $r_1, r_2, \ldots, r_n$, the polynomial is:
   $$f(x) = a(x - r_1)(x - r_2)\ldots(x - r_n)$$
   where a is a constant.

—

Marking Guidelines

Marking Scheme

Problem 16.1: Finding the Sum and Product of Roots

- Correct application of formulas for sum and product [2 marks per part]
- Accurate verification of results [2 marks per part]
- Valid explanation of the formulas [2 marks]

Problem 16.2: Finding a Polynomial Whose Roots Are a Function of the Original Roots

- Correct transformation of roots [2 marks per part]
- Accurate construction of the new polynomial [2 marks per part]
- Valid explanation of the process [2 marks]

Problem 16.3: Finding a Polynomial with Given Roots

- Correct construction of the polynomial [2 marks per part]
- Accurate use of the sum and product of roots [2 marks per part]
- Valid explanation of the general form [2 marks]

17 More Rational Functions

Problem 17.1: Rational Functions of the Form $f(x) = \frac{ax+b}{cx^2+dx+e}$

Problem Statement

1. For each of the following rational functions, find the equations of the vertical and horizontal asymptotes, and sketch the graph:

1. $f(x) = \frac{2x+3}{x^2-4}$
2. $g(x) = \frac{x-1}{x^2+x-6}$
3. $h(x) = \frac{3x+2}{2x^2-5x+3}$

2. Explain why the horizontal asymptote of $f(x) = \frac{ax+b}{cx^2+dx+e}$ is always $y = 0$.

3. For $f(x) = \frac{2x+1}{x^2-3x+2}$:

- Find the x- and y-intercepts
- Identify the vertical asymptotes
- Sketch the graph

—

Problem 17.2: Rational Functions of the Form $f(x) = \frac{ax^2+bx+c}{dx+e}$

Problem Statement

1. For each of the following rational functions, find the equations of the vertical and oblique asymptotes, and sketch the graph:

1. $f(x) = \frac{x^2+2x+1}{x-1}$
2. $g(x) = \frac{2x^2-3x+4}{x+2}$
3. $h(x) = \frac{x^2-4}{x+1}$

2. Explain how to find the oblique asymptote of $f(x) = \frac{ax^2+bx+c}{dx+e}$ using polynomial long division.

3. For $f(x) = \frac{x^2-3x+2}{x-2}$:

- Perform polynomial long division to find the oblique asymptote
- Identify the vertical asymptote
- Sketch the graph

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Vertical Asymptotes**: Vertical asymptotes occur where the denominator of the rational function equals zero, provided the numerator does not also equal zero at the same point.

2. **Horizontal Asymptotes**: For $f(x) = \frac{ax+b}{cx^2+dx+e}$:

$$\text{Horizontal asymptote: } y = 0 \quad (\text{as } x \to \pm\infty).$$

3. **Oblique Asymptotes**: For $f(x) = \frac{ax^2+bx+c}{dx+e}$:

   - Perform polynomial long division to express $f(x)$ as:

$$f(x) = q(x) + \frac{r(x)}{dx+e}$$

   where $q(x)$ is the oblique asymptote.

4. **Steps to Sketch the Graph**:

   (a) Find the vertical asymptotes by solving $dx + e = 0$.

   (b) Find the horizontal or oblique asymptotes.

   (c) Determine the x- and y-intercepts.

   (d) Analyze the behavior of the function near the asymptotes.

   (e) Sketch the graph, ensuring it approaches the asymptotes correctly.

—

Marking Guidelines

Marking Scheme

Problem 17.1: Rational Functions of the Form $f(x) = \frac{ax+b}{cx^2+dx+e}$

- Correct identification of vertical asymptotes [2 marks per part]
- Accurate determination of horizontal asymptotes [2 marks per part]
- Accurate graph sketches [3 marks per graph]
- Valid explanation of horizontal asymptotes [2 marks]

Problem 17.2: Rational Functions of the Form $f(x) = \frac{ax^2+bx+c}{dx+e}$

- Correct identification of vertical and oblique asymptotes [2 marks per part]
- Accurate use of polynomial long division [2 marks per part]
- Accurate graph sketches [3 marks per graph]
- Valid explanation of oblique asymptotes [2 marks]

Additional Points

- Clear labeling of graphs and asymptotes [1 mark per graph]
- Logical presentation of solutions [1 mark]

18 Properties of Functions

Problem 18.1: Odd and Even Functions

Problem Statement

1. Determine algebraically whether the following functions are odd, even, or neither:

1. $f(x) = x^2$
2. $g(x) = x^3$
3. $h(x) = x^2 + x$
4. $k(x) = \sin(x)$

2. Sketch the graphs of the following functions and determine graphically whether they are odd, even, or neither:

1. $f(x) = |x|$
2. $g(x) = x^3 - x$
3. $h(x) = \cos(x)$

3. Explain the conditions for a function to be:

- Even: $f(-x) = f(x)$
- Odd: $f(-x) = -f(x)$

—

Problem 18.2: Finding the Inverse Function and Domain Restriction

Problem Statement

1. Find the inverse of the following functions and state the largest possible domain for which the inverse exists:

1. $f(x) = x^2$
2. $g(x) = e^x$
3. $h(x) = \ln(x)$

2. For $f(x) = x^2$, restrict the domain so that the inverse exists and find $f^{-1}(x)$.

3. Explain why a function must be one-to-one for its inverse to exist.

—

Problem 18.3: Self-Inverse Functions

Problem Statement

1. Determine algebraically whether the following functions are self-inverse:

1. $f(x) = \frac{1}{x}$
2. $g(x) = x$
3. $h(x) = 2x - 3$

2. Sketch the graphs of the following functions and determine graphically whether they are self-inverse:

1. $f(x) = \frac{1}{x}$
2. $g(x) = x$
3. $h(x) = 2x - 3$

3. Explain the conditions for a function to be self-inverse:

- Algebraically: $f(f(x)) = x$
- Graphically: The graph is symmetric about the line $y = x$

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Odd and Even Functions**:
   - A function is even if $f(-x) = f(x)$ for all x in the domain.
   - A function is odd if $f(-x) = -f(x)$ for all x in the domain.
2. **Inverse Function**:
   - The inverse of $f(x)$, denoted $f^{-1}(x)$, satisfies:

     $$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

   - A function must be one-to-one (pass the horizontal line test) for its inverse to exist.
3. **Self-Inverse Function**:
   - A function is self-inverse if $f(f(x)) = x$ for all x in the domain.
   - Graphically, the function is symmetric about the line $y = x$.

—

Marking Guidelines

Marking Scheme

Problem 18.1: Odd and Even Functions

- Correct algebraic determination of odd, even, or neither [2 marks per part]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of conditions for odd and even functions [2 marks]

Problem 18.2: Finding the Inverse Function and Domain Restriction

- Correct calculation of the inverse function [2 marks per part]
- Accurate identification of the largest possible domain [2 marks per part]
- Valid explanation of the one-to-one condition [2 marks]

Problem 18.3: Self-Inverse Functions

- Correct algebraic determination of self-inverse property [2 marks per part]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of self-inverse conditions [2 marks]

19 Inequalities

Problem 19.1: Solving Cubic Inequalities Without Technology

Problem Statement

1. Solve the following cubic inequalities algebraically:

1. $(x-1)(x+2)(x-3) \geq 0$
2. $x^3 - 6x^2 + 11x - 6 < 0$
3. $(x-2)^2(x+1) \leq 0$

2. For the inequality $x^3 - x^2 - 4x + 4 > 0$:

- Factor the cubic expression
- Create a sign diagram
- State the solution

3. Explain the method of solving cubic inequalities using:

- Factorization
- Sign diagrams
- Critical points

—

Problem 19.2: Solving Inequalities Graphically with Technology

Problem Statement

1. Using your GDC, solve the following inequalities by graphing both sides:

1. $x^2 + 2x - 3 > x + 1$
2. $\frac{x}{x-2} \leq 3$
3. $|x - 1| < x^2 - 2x + 2$

2. Solve the system of inequalities graphically:

$$x^2 - 4 \geq 0$$
$$x^2 - 2x - 3 < 0$$

3. For the inequality $\frac{x+1}{x-2} > x$:

- Graph both sides
- Identify any vertical asymptotes
- State the solution

—

Problem 19.3: Mixed Inequality Problems

Problem Statement

1. Solve the following inequalities using the most appropriate method (algebraic or graphical):

1. $(x^2 - 4)(x + 1) > 0$
2. $\frac{x^2+x-2}{x-1} \leq 0$
3. $|x^2 - 4| > 2x$

2. For each solution in part 1:

- Verify your solution using the alternative method
- Explain which method was more efficient and why

3. Create a step-by-step guide for deciding whether to use algebraic or graphical methods for solving inequalities.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Solving Cubic Inequalities Algebraically**:
   - Factor the cubic expression
   - Identify critical points (zeros)
   - Create a sign diagram
   - Read the solution from the sign diagram
2. **Solving Inequalities Graphically**:
   - Graph $y = f(x)$ and $y = g(x)$
   - For $f(x) > g(x)$: Find where $f(x)$ is above $g(x)$
   - For $f(x) < g(x)$: Find where $f(x)$ is below $g(x)$
   - Consider domain restrictions and asymptotes
3. **Special Cases**:
   - Rational inequalities: Consider domain restrictions
   - Absolute value inequalities: Consider both cases
   - Multiple inequalities: Use intersection or union

—

Marking Guidelines

Marking Scheme

Problem 19.1: Solving Cubic Inequalities Without Technology

- Correct factorization [2 marks per part]
- Accurate sign diagram [2 marks per part]
- Valid solution with correct interval notation [2 marks per part]

Problem 19.2: Solving Inequalities Graphically with Technology

- Correct graphs [2 marks per part]
- Accurate identification of solution regions [2 marks per part]
- Valid consideration of domain restrictions and asymptotes [2 marks per part]

Problem 19.3: Mixed Inequality Problems

- Correct choice of method [1 mark per part]
- Accurate solution [2 marks per part]
- Valid verification using alternative method [2 marks per part]
- Clear explanation of method choice [2 marks]

Additional Points

- Clear presentation of solutions [1 mark]
- Correct use of interval notation [1 mark]
- Logical reasoning in method selection [1 mark]

20 The Modulus Function

Problem 20.1: Sketching Graphs of the Form $y = |f(x)|$

Problem Statement

1. Sketch the graphs of the following functions:

1. $y = |x|$
2. $y = |x^2 - 4|$
3. $y = |x - 2| + 1$

2. For $f(x) = x^3 - 3x^2 + 2x$, sketch the graph of $y = |f(x)|$.

3. Explain how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$.

—

Problem 20.2: Sketching Graphs of the Form $y = f(|x|)$

Problem Statement

1. Sketch the graphs of the following functions:

1. $y = (|x|)^2$
2. $y = \sin(|x|)$
3. $y = e^{-|x|}$

2. For $f(x) = x^2 - 4x + 3$, sketch the graph of $y = f(|x|)$.

3. Explain how the graph of $y = f(|x|)$ is obtained from the graph of $y = f(x)$.

—

Problem 20.3: Solving Equations Involving the Modulus Function

Problem Statement

1. Solve the following equations involving the modulus function:

1. $|x-3|=5$
2. $|2x+1|=7$
3. $|x^2-4|=3$

2. Solve $|x-2|=|x+1|$ and interpret the solution graphically.

3. Explain the general method for solving equations of the form $|f(x)|=g(x)$.

—

Problem 20.4: Solving Inequalities Involving the Modulus Function

Problem Statement

1. Solve the following inequalities involving the modulus function:

1. $|x-3|\leq 5$
2. $|2x+1|>7$
3. $|x^2-4|<3$

2. Solve $|x-2|\leq|x+1|$ and interpret the solution graphically.

3. Explain the general method for solving inequalities of the form $|f(x)|\leq g(x)$ or $|f(x)|>g(x)$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Graph of $y = |f(x)|$**:
   - Reflect the portion of the graph of $y = f(x)$ that lies below the x-axis above the x-axis.
   - The graph of $y = |f(x)|$ is always non-negative.
2. **Graph of $y = f(|x|)$**:
   - Replace the portion of the graph of $y = f(x)$ for $x < 0$ with a reflection of the portion for $x > 0$.
   - The graph of $y = f(|x|)$ is symmetric about the y-axis.
3. **Solving Modulus Equations**:
   - For $|f(x)| = g(x)$:
     $$f(x) = g(x) \quad \text{or} \quad f(x) = -g(x)$$
   - Solve each case separately and check for extraneous solutions.
4. **Solving Modulus Inequalities**:
   - For $|f(x)| \leq g(x)$:
     $$-g(x) \leq f(x) \leq g(x)$$
   - For $|f(x)| > g(x)$:
     $$f(x) > g(x) \quad \text{or} \quad f(x) < -g(x)$$
   - Solve each case separately and check for extraneous solutions.

—

Marking Guidelines

Marking Scheme

Problem 20.1: Sketching Graphs of the Form $y = |f(x)|$

- Correct reflection of the negative portion of the graph [2 marks per graph]
- Accurate labeling of key points [1 mark per graph]
- Valid explanation of the transformation [2 marks]

Problem 20.2: Sketching Graphs of the Form $y = f(|x|)$

- Correct reflection of the graph for $x < 0$ [2 marks per graph]
- Accurate labeling of key points [1 mark per graph]
- Valid explanation of the transformation [2 marks]

Problem 20.3: Solving Modulus Equations

- Correct splitting of the modulus equation into cases [2 marks per part]
- Accurate solutions for each case [2 marks per part]
- Valid interpretation of the solution graphically [2 marks]

Problem 20.4: Solving Modulus Inequalities

- Correct splitting of the modulus inequality into cases [2 marks per part]
- Accurate solutions for each case [2 marks per part]
- Valid interpretation of the solution graphically [2 marks]

Additional Points

- Clear presentation of solutions [1 mark]
- Correct use of interval notation [1 mark]
- Logical reasoning in solving equations and inequalities [1 mark]

21 More Transformations of Graphs

Problem 21.1: Sketching Graphs of the Form $y = \frac{1}{f(x)}$

Problem Statement

1. Sketch the graphs of the following functions:

1. $y = \frac{1}{x}$
2. $y = \frac{1}{x^2-4}$
3. $y = \frac{1}{\sin(x)}$ for $x \in [0, 2\pi]$

2. For $f(x) = x^2 - 4$, sketch the graph of $y = \frac{1}{f(x)}$ and:

- Identify the vertical asymptotes
- Determine the behavior of the graph near the asymptotes
- State the domain and range

3. Explain how the graph of $y = \frac{1}{f(x)}$ is related to the graph of $y = f(x)$.

—

Problem 21.2: Sketching Graphs of the Form $y = f(ax + b)$

Problem Statement

1. Sketch the graphs of the following functions:

1. $y = f(2x)$, where $f(x) = x^2$
2. $y = f(x - 3)$, where $f(x) = \sqrt{x}$
3. $y = f(2x + 1)$, where $f(x) = \sin(x)$ for $x \in [0, 2\pi]$

2. For $f(x) = x^3$, sketch the graph of $y = f(3x - 2)$ and:

- Describe the horizontal stretch/compression
- Describe the horizontal translation
- State the domain and range

3. Explain how the graph of $y = f(ax + b)$ is obtained from the graph of $y = f(x)$.

—

Problem 21.3: Sketching Graphs of the Form $y = [f(x)]^2$

Problem Statement

1. Sketch the graphs of the following functions:

1. $y = [x]^2$
2. $y = [x^2 - 4]^2$
3. $y = [\sin(x)]^2$ for $x \in [0, 2\pi]$

2. For $f(x) = x^2 - 4$, sketch the graph of $y = [f(x)]^2$ and:

- Identify the x-intercepts
- Describe the behavior of the graph near the x-intercepts
- State the domain and range

3. Explain how the graph of $y = [f(x)]^2$ is related to the graph of $y = f(x)$.

—

Key Formulas and Definitions

Key Formulas and Definitions

1. **Graph of $y = \frac{1}{f(x)}$**:
   - Vertical asymptotes occur where $f(x) = 0$.
   - The graph approaches $y = 0$ as $|f(x)| \to \infty$.
   - The graph has the same sign as $f(x)$.
2. **Graph of $y = f(ax + b)$**:
   - Horizontal stretch/compression by a factor of $\frac{1}{|a|}$.
   - Horizontal translation by $-\frac{b}{a}$.
3. **Graph of $y = [f(x)]^2$**:
   - The graph is always non-negative.
   - The x-intercepts of $f(x)$ remain the same.
   - The graph has a "U-shape" near the x-intercepts of $f(x)$.

—

Marking Guidelines

Marking Scheme

Problem 21.1: Sketching Graphs of the Form $y = \frac{1}{f(x)}$

- Correct identification of vertical asymptotes [2 marks per graph]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of the transformation [2 marks]

Problem 21.2: Sketching Graphs of the Form $y = f(ax + b)$

- Correct description of horizontal stretch/compression and translation [2 marks per graph]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of the transformation [2 marks]

Problem 21.3: Sketching Graphs of the Form $y = [f(x)]^2$

- Correct identification of x-intercepts and behavior near them [2 marks per graph]
- Accurate graph sketches [2 marks per graph]
- Valid explanation of the transformation [2 marks]

Additional Points

- Clear labeling of graphs and key points [1 mark per graph]
- Logical reasoning in transformations [1 mark]

Conclusion

Mathematics is not just about understanding theory; it is about applying concepts to solve problems effectively. This guide has provided you with a collection of expertly crafted practice problems focused on Functions, designed to challenge your understanding and enhance your problem-solving skills.
For detailed solutions and answers, keep following me — they will be available soon! If you're looking for personalized guidance, book a one-on-one mentorship session with me to deepen your understanding of IB Mathematics AA/AI HL, Functions, or even Olympiad-level problems. Together, we can build the confidence and skills you need to excel in mathematics.

As you prepare for your exams, remember:

- **Practice is the key to success**: The more problems you solve, the more confident and efficient you become. Focus on understanding the logic behind each solution rather than memorizing formulas.
- **Learn from mistakes**: Every mistake is an opportunity to grow. Analyze where you went wrong and refine your approach.
- **Time management is crucial**: Simulate exam conditions to improve your speed and accuracy under pressure.

If you're aiming for a guaranteed improvement and want to elevate your performance to the next level, consider applying for my **exclusive personalized mentorship program**. As an alumnus of IIT Guwahati and ISI, with over 5 years of teaching experience from the school level to university students, now mentoring high-achieving IB students, I specialize in:

- **Tailored guidance**: Customized study plans and strategies based on your strengths and weaknesses.
- **Exam-focused preparation**: Insights into examiner expectations and tips to maximize your score.
- **Beyond IB HL Problem-Solving**: My mentorship is not limited to IB HL Mathematics. I will enrich your mathematical thinking to push you toward **Olympiad-level problem-solving** and help you excel in **quantitative aptitude**, preparing you for competitive exams and real-world challenges.
- **One-on-one mentorship**: Direct support to clarify doubts, build confidence, and achieve your goals.

Join the ranks of students who have transformed their performance and achieved top scores with my mentorship. Visit www.mathematicselevateacademy.com to access free resources, book a session, or apply for the program. Let's work together to make your IB Mathematics journey a success!

"Success in mathematics comes not from the number of problems you've solved, but from the confidence you've gained in solving them."

- Rishabh Kumar
Founder, Mathematics Elevate Academy
Elite Mentor for IB Mathematics
Alumnus of IIT Guwahati & Indian Statistical Institute

Thank You!

Rishabh Kumar
Mathematics Elevate Academy
www.mathematicselevateacademy.com