International Baccalaureate (IB) Diploma Programme Mathematics Analysis and Approaches Higher Level

Calculus

The IB 7-Scorer's Ultimate Guide

Crafted Exclusively for High-Achieving IB Mathematics Students: April 2025 Edition

Mathematics Elevate Academy

Excellence in Advanced Mathematics Education

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Disclaimer

Introduction

The IB 7-Scorer's Ultimate Guide — April 2025 Edition is thoughtfully designed for IB DP Mathematics students striving to excel in IB DP Mathematics AA SL/HL, with a special emphasis on calculus and conceptual mastery. This guide offers a wealth of expertly crafted high-level calculus problems, conceptual challenges, and much more.

Explore examiner-style solutions, detailed marking scheme breakdowns, and insightful commentary on common errors to refine your problem-solving skills. Each problem is designed to test your grasp of calculus concepts, from differentiation and integration to advanced applications like optimization, differential equations, and area/volume calculations.

This guide goes beyond the IB syllabus, offering enrichment problems that challenge your mathematical thinking and prepare you for Olympiads and university-level mathematics. The solutions are presented with step-by-step clarity, expert insights, and advanced techniques, ensuring a comprehensive and engaging learning experience.

For answers or detailed solutions, keep following me — they will be available soon! For personalized learning, book a one-on-one mentorship session with me to receive customized guidance on mastering IB DP Mathematics AA/AI HL, calculus, or even Olympiad-level problems. Together, we will build the confidence and skills you need to excel.

Check Your Understanding!

Standard Level

1 Concepts of a Limit and Derivative

Problem 1.1: Limits and Derivatives

Problem Statement

1. True/False: If the limit of a function $f(x)$ as $x \to a$ exists, then $f(x)$ must be continuous at $x = a$. Justify your answer.

2. Estimate the value of the following limit using the table of values provided:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

x	$\frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
2.001	4.001
2.01	4.01
2.1	4.1

3. Sketch the graph of $f(x) = \frac{x^2-4}{x-2}$ and use it to verify your result from part (2).

Problem 1.2: Derivatives as Gradients and Rates of Change

Problem Statement

A particle moves along a straight line, and its position at time t (in seconds) is given by the function:

$$s(t) = t^3 - 6t^2 + 9t$$

1. Find the velocity function $v(t)$ of the particle by differentiating $s(t)$.
2. Determine the time(s) at which the particle is at rest.
3. Interpret the derivative $v(t)$ as the rate of change of position and explain what it means when $v(t) = 0$.
4. Estimate the instantaneous velocity of the particle at $t = 1$ by calculating the gradient of the chord between $t = 1$ and $t = 1.01$.

Problem 1.3: Gradient as a Limit of Chords

Problem Statement

The function $f(x) = x^2 + 2x$ is given.

1. Write the formula for the gradient of the chord between $x = 1$ and $x = 1 + h$.
2. Simplify the formula and find the gradient as $h \to 0$.
3. Verify that the result matches the derivative $f'(x)$ at $x = 1$.

Formula to Use

Key Formulas

1. The derivative of a function $f(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. The velocity of a particle is the derivative of its position function:

$$v(t) = \frac{ds}{dt}$$

3. The gradient of a chord between $x = a$ and $x = a + h$ is:

$$\text{Gradient} = \frac{f(a+h) - f(a)}{h}$$

Marking Guidelines

Marking Scheme

Problem 1: Limits and Derivatives

- Correct identification and justification of True/False [4 marks]
- Accurate limit estimation from table values [3 marks]
- Correct graph sketch with hole at $x = 2$ [3 marks]
- Valid verification using graph [2 marks]

Problem 2: Derivatives as Gradients and Rates of Change

- Correct velocity function derivation [2 marks]
- Finding times when particle is at rest [3 marks]
- Clear interpretation of $v(t)$ and $v(t) = 0$ [2 marks]
- Accurate instantaneous velocity calculation [4 marks]

Problem 3: Gradient as a Limit of Chords

- Correct gradient formula setup [2 marks]
- Accurate simplification and limit evaluation [3 marks]
- Valid verification with $f'(x)$ [2 marks]
- Clear presentation of working [1 mark]

2 Increasing and Decreasing Functions

Problem 2.1: Function Behavior Analysis

Problem Statement

Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$

1. Find $f'(x)$.
2. Solve $f'(x) = 0$ to find critical points.
3. Create a sign diagram for $f'(x)$ and use it to determine intervals where:
   - $f(x)$ is increasing
   - $f(x)$ is decreasing
4. Verify your answers by sketching $f(x)$.

Problem 2.2: Derivative Graph Analysis

Problem Statement

The graph of $f'(x)$, the derivative of a function $f(x)$, is shown below:
[Insert a graph showing a cubic function for $f'(x)$ crossing the x-axis at x = -1, 1, and 3]

1. Identify all intervals where $f(x)$ is:
   - Increasing
   - Decreasing
2. Locate and classify all local extrema of $f(x)$.
3. Sketch a possible graph of $f(x)$ that matches your analysis.
4. If $f(0) = 2$, determine whether $f(4)$ is greater than or less than 2. Justify your answer.

Problem 2.3: Advanced Function Reconstruction

Problem Statement

The derivative $f'(x)$ of a function $f(x)$ is given by:

$$f'(x) = (x+2)(x-1)^2$$

1. Find all values of x where $f'(x) = 0$.
2. Determine the intervals where $f(x)$ is increasing and decreasing.
3. Given that $f(0) = 3$:
   - Sketch a possible graph of $f(x)$
   - Explain why your sketch is not unique
   - Find $f(2) - f(-2)$ using the fundamental theorem of calculus

Key Concepts and Formulas

Important Relationships

1. For a function $f(x)$:
   - If $f'(x) > 0$, then $f(x)$ is increasing
   - If $f'(x) < 0$, then $f(x)$ is decreasing
   - If $f'(x) = 0$, then x is a critical point
2. The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

3. At local extrema:
   - Local maximum: $f'(x)$ changes from positive to negative
   - Local minimum: $f'(x)$ changes from negative to positive

Marking Guidelines

Marking Scheme

Problem 1:

- Correct derivative calculation [2 marks]
- Finding critical points [3 marks]
- Complete sign diagram [3 marks]
- Accurate graph with justification [4 marks]

Problem 2:

- Correct interval identification [3 marks]
- Accurate extrema classification [3 marks]
- Valid function sketch [4 marks]
- Correct comparison with justification [2 marks]

Problem 3:

- Finding zeros of $f'(x)$ [3 marks]
- Interval analysis [3 marks]
- Valid sketch with explanation [3 marks]
- Correct definite integral calculation [3 marks]

3 Derivatives of Polynomials

Problem 3.1: Differentiating Polynomials

Problem Statement

Consider the polynomial function:

$$f(x) = 3x^5 - 4x^3 + 7x^2 - 2x + 5$$

1. Find $f'(x)$, the derivative of $f(x)$.
2. Evaluate $f'(x)$ at $x = 2$.
3. Determine the equation of the tangent line to $f(x)$ at $x = 2$.

Problem 3.2: Rearranging and Differentiating

Problem Statement

Simplify the following expression into the form $f(x) = ax^n + bx^{n-1} + \ldots$, and then differentiate it:

$$f(x) = \frac{2x^3}{x} + 5x^2 - \frac{3}{x}$$

1. Simplify $f(x)$ into a single polynomial expression.
2. Find $f'(x)$.
3. Evaluate $f'(x)$ at $x = 1$.

Problem 3.3: Higher-Order Derivatives

Problem Statement

Let $f(x) = x^4 - 6x^3 + 11x^2 - 6x$.

1. Find the first derivative $f'(x)$.
2. Find the second derivative $f''(x)$.
3. Determine the values of x where $f''(x) = 0$.
4. Explain the significance of $f''(x) = 0$ in terms of the graph of $f(x)$.

Problem 3.4: Polynomial Application in Motion

Problem Statement

The position of a particle moving along a straight line is given by:

$$s(t) = t^3 - 6t^2 + 9t + 4$$

1. Find the velocity function $v(t)$ by differentiating $s(t)$.
2. Find the acceleration function $a(t)$ by differentiating $v(t)$.
3. Determine the time(s) at which the particle is at rest.
4. Determine whether the particle is speeding up or slowing down at $t = 2$.

Key Concepts and Formulas

Important Relationships

1. The derivative of a polynomial $f(x) = ax^n$ is:

$$f'(x) = n \cdot ax^{n-1}$$

2. The second derivative $f''(x)$ provides information about the concavity of $f(x)$:

   - $f''(x) > 0$: $f(x)$ is concave up.
   - $f''(x) < 0$: $f(x)$ is concave down.
   - $f''(x) = 0$: Possible inflection point.

3. For motion:

   - Velocity: $v(t) = \frac{ds}{dt}$
   - Acceleration: $a(t) = \frac{dv}{dt}$

Marking Guidelines

Marking Scheme

Problem 1:

- Correct derivative calculation [2 marks]
- Correct evaluation of $f'(x)$ at $x = 2$ [2 marks]
- Correct tangent line equation [3 marks]

Problem 2:

- Correct simplification of $f(x)$ [2 marks]
- Correct derivative calculation [2 marks]
- Correct evaluation of $f'(x)$ at $x = 1$ [2 marks]

Problem 3:

- Correct first derivative [2 marks]
- Correct second derivative [2 marks]
- Correct solution for $f''(x) = 0$ [2 marks]
- Explanation of significance of $f''(x) = 0$ [2 marks]

Problem 4:

- Correct velocity function [2 marks]
- Correct acceleration function [2 marks]
- Correct solution for when the particle is at rest [2 marks]
- Correct analysis of speeding up/slowing down [2 marks]

4 Equations of Tangents and Normals

Problem 4.1: Gradient Evaluation and Tangent Lines

Problem Statement

Consider the curve $y = x^3 - 3x^2 + 2$.

1. Find $\frac{dy}{dx}$.
2. Evaluate the gradient at the point where $x = 2$.
3. Find the coordinates of the point on the curve where $x = 2$.
4. Find the equation of the tangent line at this point using:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is the point and m is the gradient.

Problem 4.2: Points with Given Gradient

Problem Statement

For the curve $y = x^2 - 4x + 5$:

1. Find all points on the curve where the gradient is 3.
2. For each point found in part (a):
   - Write the equation of the tangent line
   - Write the equation of the normal line using:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

3. Verify your answers using a graphing calculator.

Problem 4.3: Advanced Applications

Problem Statement

The curve $y = \frac{1}{3}x^3 - 2x + 1$ is given.

1. Find the coordinates of the point(s) where the tangent line is parallel to the line $y = 2x + 1$.
2. Find the equation of the normal line at the point where $x = 1$.
3. Determine whether there are any points on the curve where the tangent line is perpendicular to the line $y = x - 3$.
4. Use technology to graph the function and verify your answers.

Problem 4.4: Technology-Based Investigation

Problem Statement

Using appropriate technology (graphing calculator or software):

1. Graph $f(x) = x^3 - 6x^2 + 9x + 1$
2. Find and plot $f'(x)$
3. Use the graphs to:
   - Identify all points where the tangent line is horizontal
   - Find the coordinates of any inflection points
   - Determine intervals where the function is increasing/decreasing
4. Verify your findings algebraically

Key Formulas and Concepts

Important Relationships

1. Equation of tangent line:
$$y - y_1 = m(x - x_1)$$
where $m = f'(x_1)$
2. Equation of normal line:
$$y - y_1 = -\frac{1}{m}(x - x_1)$$
3. Parallel lines have equal gradients
4. Perpendicular lines have gradients that are negative reciprocals:
$$m_1 \cdot m_2 = -1$$
5. Horizontal tangents occur when $f'(x) = 0$
6. Vertical tangents occur when $f'(x)$ is undefined

Marking Guidelines

Marking Scheme

Problem 1:

- Correct derivative [2 marks]
- Correct gradient evaluation [2 marks]
- Correct point coordinates [2 marks]
- Correct tangent equation [3 marks]

Problem 2:

- Finding points with gradient 3 [3 marks]
- Correct tangent equations [3 marks]
- Correct normal equations [3 marks]
- Proper verification [2 marks]

Problem 3:

- Finding parallel tangent points [3 marks]
- Correct normal equation [3 marks]
- Analysis of perpendicular tangents [3 marks]
- Proper verification [2 marks]

Problem 4:

- Correct graphs [2 marks]
- Identification of key points [3 marks]
- Correct interval analysis [3 marks]
- Algebraic verification [3 marks]

5 Introduction to Integration

Problem 5.1: Basic Integration as Anti-Differentiation

Problem Statement

Evaluate the following indefinite integrals:

1. $\int (3x^4 - 2x^3 + 5x^2)\,dx$
2. $\int (4x^3 - 6x + 7)\,dx$
3. $\int (2x^{-3} + 5x^{-2})\,dx$

Key Formula: For $f(x) = ax^n$, where $n \neq -1$:

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C$$

Problem 5.2: Rearranging Before Integration

Problem Statement

Simplify the following expressions into the form $f(x) = ax^n + bx^{n-1} + \ldots$ before integrating:

1. $\int \left(\frac{2x^3}{x} + 5x^2 - \frac{3}{x}\right)\,dx$
2. $\int \left(\frac{6x^2}{x^3} - 4x^{-2} + 7\right)\,dx$

Problem 5.3: Definite Integrals and Area Under a Curve

Problem Statement

Use definite integration to find the area under the curve $y = x^2 - 2x + 3$ between $x = 1$ and $x = 4$.

1. Evaluate $\int_1^4 (x^2 - 2x + 3)\,dx$.
2. Interpret the result as the area between the curve and the x-axis.
3. Verify your result using technology.

Problem 5.4: Anti-Differentiation with Boundary Conditions

Problem Statement

Find the expression for y in terms of x given the following differential equations and boundary conditions:

1. $\frac{dy}{dx} = 3x^2 - 4x + 5$, and $y = 10$ when $x = 1$.
2. $\frac{dy}{dx} = 2x^{-2} + 6x$, and $y = 4$ when $x = 2$.

Key Formula: For $\frac{dy}{dx} = f(x)$:

$$y = \int f(x)\,dx + C$$

Use the boundary condition to solve for C.

Problem 5.5: Area of a Region Enclosed by a Curve

Problem Statement

The curve $y = x^3 - 3x^2 + 4$ intersects the x-axis at $x = 0$ and $x = 2$. Find the area of the region enclosed by the curve and the x-axis between $x = 0$ and $x = 2$.

1. Set up the definite integral $\int_0^2 (x^3 - 3x^2 + 4)\,dx$.
2. Evaluate the integral and interpret the result.
3. Verify your result using technology.

Key Concepts and Formulas

Important Relationships

1. Indefinite integration (anti-differentiation):

$$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + C, \quad n \neq -1$$

2. Definite integration:

$$\int_a^b f(x)\, dx = F(b) - F(a), \quad \text{where } F(x) \text{ is the anti-derivative of } f(x)$$

3. Area under a curve:

$$\text{Area} = \int_a^b f(x)\, dx, \quad \text{where } f(x) > 0 \text{ on } [a, b]$$

4. Anti-differentiation with boundary conditions:

$$y = \int f(x)\, dx + C, \quad \text{where } C \text{ is determined using the boundary condition.}$$

Marking Guidelines

Marking Scheme

Problem 1:

- Correct anti-derivative for each term [2 marks per integral]
- Proper use of the constant of integration C [1 mark]

Problem 2:

- Correct simplification of the expression [2 marks per integral]
- Correct anti-derivative [2 marks per integral]

Problem 3:

- Correct setup of the definite integral [2 marks]
- Correct evaluation of the integral [3 marks]
- Proper interpretation of the result as an area [2 marks]

Problem 4:

- Correct anti-derivative [2 marks per equation]
- Correct use of the boundary condition to find C [2 marks per equation]
- Final expression for y [1 mark per equation]

Problem 5:

- Correct setup of the definite integral [2 marks]
- Correct evaluation of the integral [3 marks]
- Proper interpretation of the result as an enclosed area [2 marks]

6 Further Differentiation

Problem 1: Differentiation of Basic Functions

Problem Statement

Differentiate the following functions with respect to x:

1. $f(x) = x^5 + 3x^2 - 7$
2. $g(x) = \sin x + 2\cos x - 5e^x$
3. $h(x) = \ln x + 4x^3 - e^x$

Key Formulae:

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Problem 2: Gradient at a Given Point

Problem Statement

For the function $f(x) = x^3 - 4x^2 + 6x - 2$:

1. Find $f'(x)$.
2. Evaluate the gradient of the curve at $x = 2$.
3. Find the equation of the tangent line to the curve at $x = 2$.

Problem 3: Chain Rule for Composite Functions

Problem Statement

Differentiate the following functions using the chain rule:

1. $y = (3x^2 + 2)^5$
2. $y = \sin(2x^3 - 4x)$
3. $y = e^{x^2+3x}$

Key Formula: For $y = g(u)$ where $u = f(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Problem 4: Product Rule

Problem Statement

Differentiate the following functions using the product rule:

1. $y = x^2 \sin x$
2. $y = e^x \ln x$
3. $y = (x^3 - 2x)(\cos x)$

Key Formula: For $y = uv$:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Problem 5: Quotient Rule

Problem Statement

Differentiate the following functions using the quotient rule:

1. $y = \frac{x^2}{\sin x}$
2. $y = \frac{\ln x}{x^2}$
3. $y = \frac{e^x}{x^3-1}$

Key Formula: For $y = \frac{u}{v}$:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Problem 6: Mixed Applications

Problem Statement

Solve the following problems involving mixed differentiation techniques:

1. Find the derivative of $y = \frac{(x^2+1)^3}{\sin x}$.
2. Differentiate $y = e^{x^2} \ln(x^3 + 1)$.
3. For $y = (x^2 + 1)(\ln x)$, find the gradient of the curve at $x = 1$.

Key Concepts and Formulas

Important Relationships

1. Basic derivatives:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

2. Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

3. Product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

4. Quotient rule:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Marking Guidelines

Marking Scheme

Problem 1:

- Correct differentiation of each term [2 marks per function]

Problem 2:

- Correct derivative [2 marks]
- Correct gradient evaluation [2 marks]
- Correct tangent equation [3 marks]

Problem 3:

- Correct application of the chain rule [3 marks per function]

Problem 4:

- Correct application of the product rule [3 marks per function]

Problem 5:

- Correct application of the quotient rule [3 marks per function]

Problem 6:

- Correct setup of the derivative [3 marks per function]
- Correct evaluation of the gradient (if applicable) [2 marks]

7 Second Derivative

Problem 1: Finding the Second Derivative

Problem Statement

Find the second derivative $f''(x)$ for the following functions:

1. $f(x) = x^4 - 3x^3 + 2x^2 - 5x + 7$
2. $f(x) = \sin x + 2\cos x$
3. $f(x) = e^{2x} - \ln x$

Key Formulae:

- First derivative: $f'(x)$
- Second derivative: $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(f'(x))$

Problem 2: Graphical Behavior of Functions

Problem Statement

The graph of $y = f(x)$ is shown below:
[Insert a graph of a cubic function, e.g., $f(x) = x^3 - 3x^2 + 2x$]

1. Sketch the graph of $y = f'(x)$, the first derivative.
2. Sketch the graph of $y = f''(x)$, the second derivative.
3. Identify the points where $f'(x) = 0$ and $f''(x) = 0$ on the graph of $f(x)$.

Problem 3: Concavity and the Second Derivative

Problem Statement

For the function $f(x) = x^3 - 6x^2 + 9x + 1$:

1. Find $f'(x)$ and $f''(x)$.
2. Determine the intervals where the graph of $f(x)$ is:
   - Concave up ($f''(x) > 0$)
   - Concave down ($f''(x) < 0$)
3. Identify any inflection points (where $f''(x) = 0$ and the concavity changes).
4. Verify your results by sketching the graph of $f(x)$.

Problem 4: Applications of the Second Derivative

Problem Statement

A particle moves along a straight line, and its position at time t is given by $s(t) = t^4 - 4t^3 + 6t^2$.

1. Find the velocity $v(t)$ and acceleration $a(t)$ of the particle.
2. Determine the time intervals where the particle's motion is:
   - Speeding up
   - Slowing down
3. Identify any points of inflection in the particle's position graph.

Problem 5: Sketching the Second Derivative from a Graph

Problem Statement

The graph of $y = f(x)$ is shown below:
[Insert a graph of a sinusoidal function, e.g., $f(x) = \sin x$]

1. Sketch the graph of $y = f'(x)$.
2. Sketch the graph of $y = f''(x)$.
3. Describe the sections of the graph of $f(x)$ as "concave up" or "concave down."

Key Concepts and Formulas

Important Relationships

1. The second derivative $f''(x)$ is the derivative of the first derivative $f'(x)$:
$$f''(x) = \frac{d}{dx}(f'(x))$$
2. Concavity:
   - $f''(x) > 0$: The graph of $f(x)$ is concave up.
   - $f''(x) < 0$: The graph of $f(x)$ is concave down.
3. Inflection points occur where $f''(x) = 0$ and the concavity changes.
4. Graphical relationships:
   - $f'(x) = 0$: Critical points (local maxima or minima) of $f(x)$.
   - $f''(x) = 0$: Possible inflection points of $f(x)$.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct first derivative [2 marks per function]
- Correct second derivative [2 marks per function]

Problem 2:

- Correct sketch of $f'(x)$ [3 marks]
- Correct sketch of $f''(x)$ [3 marks]
- Identification of key points [2 marks]

Problem 3:

- Correct derivatives $f'(x)$ and $f''(x)$ [3 marks]
- Correct intervals of concavity [3 marks]
- Identification of inflection points [2 marks]
- Accurate graph sketch [2 marks]

Problem 4:

- Correct velocity and acceleration [3 marks]
- Correct intervals of speeding up/slowing down [3 marks]
- Identification of inflection points [2 marks]

Problem 5:

- Correct sketch of $f'(x)$ [3 marks]
- Correct sketch of $f''(x)$ [3 marks]
- Accurate description of concavity [2 marks]

8 Maximum, Minimum and Inflection Points

Problem 1: Local Maximum and Minimum Points

Problem Statement

For the function $f(x) = x^3 - 6x^2 + 9x + 1$:

1. Find $f'(x)$ and solve $f'(x) = 0$ to locate the critical points.
2. Use the second derivative $f''(x)$ to determine whether each critical point is a local maximum, local minimum, or neither.
3. Verify your results by sketching the graph of $f(x)$.

Problem 2: Optimization in a Real-Life Context

Problem Statement

A farmer wants to build a rectangular pen with a fixed perimeter of 100 meters. The pen is divided into two equal sections by a fence parallel to one of the sides.

1. Let the width of the pen be x meters. Write an expression for the total area of the pen in terms of x.
2. Find the value of x that maximizes the area of the pen.
3. Verify that your solution gives a maximum by using the second derivative test.
4. Calculate the maximum area of the pen.

Problem 3: Points of Inflection

Problem Statement

For the function $f(x) = x^4 - 4x^3 + 6x^2$:

1. Find $f'(x)$ and $f''(x)$.
2. Solve $f''(x) = 0$ to locate potential points of inflection.
3. Check whether the concavity changes at each point by analyzing the sign of $f''(x)$ on either side of the point.
4. Verify your results by sketching the graph of $f(x)$.

Problem 4: Mixed Applications

Problem Statement

Solve the following problems involving maximum, minimum, and inflection points:

1. For the function $f(x) = e^x - 2x$, find the local maximum or minimum points and classify them using the second derivative test.
2. For the function $f(x) = \ln(x^2 + 1)$, find the points of inflection and verify that the concavity changes at these points.
3. A company's profit (in thousands of dollars) is modeled by $P(x) = -2x^3 + 12x^2 - 20x + 5$, where x is the number of units produced (in hundreds). Find the production level that maximizes the company's profit and calculate the maximum profit.

Key Concepts and Formulas

Important Relationships

1. **Critical Points**: Solve $f'(x) = 0$ to find critical points.
2. **Second Derivative Test**:
   - If $f''(x) > 0$ at a critical point, it is a local minimum.
   - If $f''(x) < 0$ at a critical point, it is a local maximum.
   - If $f''(x) = 0$, the test is inconclusive.
3. **Points of Inflection**:
   - Solve $f''(x) = 0$ to find potential points of inflection.
   - Verify that the concavity changes (i.e., $f''(x)$ changes sign) at these points.
4. **Optimization**:
   - Write the function to be optimized in terms of a single variable.
   - Find critical points by solving $f'(x) = 0$.
   - Use the second derivative test to confirm whether the critical point is a maximum or minimum.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct first derivative and critical points [3 marks]
- Correct second derivative and classification of points [3 marks]
- Accurate graph sketch [2 marks]

Problem 2:

- Correct expression for the area [2 marks]
- Correct value of x that maximizes the area [3 marks]
- Correct second derivative test [2 marks]
- Correct maximum area calculation [2 marks]

Problem 3:

- Correct first and second derivatives [3 marks]
- Correct identification of potential inflection points [2 marks]
- Correct concavity analysis [3 marks]
- Accurate graph sketch [2 marks]

Problem 4:

- Correct derivatives and critical points for each function [3 marks per part]
- Correct classification of points (maximum, minimum, or inflection) [3 marks per part]
- Correct optimization results for the real-life context [3 marks]

9 Introduction to Kinematics

Problem 1: Differentiation to Find Velocity and Acceleration

Problem Statement

The displacement of a particle moving along a straight line is given by $s(t) = t^3 - 6t^2 + 9t + 4$, where $s(t)$ is in meters and t is in seconds.

1. Find the velocity $v(t)$ of the particle by differentiating $s(t)$.
2. Find the acceleration $a(t)$ of the particle by differentiating $v(t)$.
3. Determine the time(s) at which the particle is at rest.
4. Find the total distance travelled by the particle in the first 4 seconds.

Problem 2: Speed as the Magnitude of Velocity

Problem Statement

The velocity of a particle is given by $v(t) = t^2 - 4t + 3$, where $v(t)$ is in meters per second and t is in seconds.

1. Find the speed of the particle at $t = 1$ and $t = 3$.
2. Determine the time intervals during which the particle is moving forward (positive velocity) and backward (negative velocity).
3. Sketch the graph of $v(t)$ and $|v(t)|$ for $0 \leq t \leq 4$.

Problem 3: Integration to Find Displacement

Problem Statement

The velocity of a particle is given by $v(t) = 3t^2 - 6t + 2$, where $v(t)$ is in meters per second and t is in seconds.

1. Find the displacement of the particle from $t = 0$ to $t = 3$ by integrating $v(t)$.
2. If the particle starts at $s(0) = 5$ meters, find its position at $t = 3$.
3. Verify your result by differentiating the displacement function.

Problem 4: Integration to Find Total Distance Travelled

Problem Statement

The velocity of a particle is given by $v(t) = t^3 - 6t^2 + 9t$, where $v(t)$ is in meters per second and t is in seconds.

1. Find the total distance travelled by the particle from $t = 0$ to $t = 3$ by integrating $|v(t)|$.
2. Determine the time intervals during which the particle is moving forward and backward.
3. Verify your result by sketching the graph of $v(t)$ and calculating the areas under the curve.

Key Concepts and Formulas

Important Relationships

1. **Velocity and Acceleration**:

$$v(t) = \frac{ds}{dt}, \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

2. **Displacement**:

$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\, dt$$

3. **Total Distance**:

$$\text{Distance} = \int_{t_1}^{t_2} |v(t)|\, dt$$

4. **Speed**:

$$\text{Speed} = |v(t)|$$

5. **Forward and Backward Motion**:

   - The particle is moving forward when $v(t) > 0$.
   - The particle is moving backward when $v(t) < 0$.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct velocity and acceleration functions [3 marks]
- Correct times when the particle is at rest [2 marks]
- Correct total distance calculation [3 marks]

Problem 2:

- Correct speed calculations [2 marks]
- Correct identification of forward and backward motion intervals [3 marks]
- Accurate graph of $v(t)$ and $|v(t)|$ [3 marks]

Problem 3:

- Correct displacement calculation [3 marks]
- Correct position at $t = 3$ [2 marks]
- Verification by differentiation [2 marks]

Problem 4:

- Correct total distance calculation [3 marks]
- Correct identification of forward and backward motion intervals [2 marks]
- Accurate graph and area verification [3 marks]

10 Further Integration Techniques

Problem 1: Basic Integration Rules

Problem Statement

Evaluate the following indefinite integrals:

1. $\int (3x^4 - 2x^3 + 5x^2)\,dx$
2. $\int (\sin x - 2\cos x + e^x)\,dx$
3. $\int \frac{1}{x}\,dx$

Key Formulae:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$

Problem 2: Integration of Composites with Linear Functions

Problem Statement

Evaluate the following integrals:

1. $\int (2x+3)^4\,dx$
2. $\int e^{3x+2}\,dx$
3. $\int \sin(5x-4)\,dx$
4. $\int \frac{1}{2x+1}\,dx$

Key Formula: For $f(ax+b)$:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C, \quad \text{where } F'(x) = f(x)$$

Problem 3: Reverse Chain Rule (Integration by Inspection)

Problem Statement

Evaluate the following integrals using the reverse chain rule:

1. $\int 3x^2(x^3+1)^5\,dx$
2. $\int 2xe^{x^2}\,dx$
3. $\int \frac{1}{(x^2+1)} \cdot 2x\,dx$
4. $\int \cos(2x)\cdot\sin(2x)\,dx$

Key Formula: For $\int g'(x)f(g(x))\,dx$:

$$\int g'(x)f(g(x))\,dx = F(g(x)) + C, \quad \text{where } F'(x) = f(x)$$

Problem 4: Substitution Method

Problem Statement

Evaluate the following integrals using substitution:

1. $\int x(x^2+1)^3\,dx$ (Let $u = x^2+1$)
2. $\int e^{2x}\,dx$ (Let $u = 2x$)
3. $\int \frac{\sin x}{\cos^2 x}\,dx$ (Let $u = \cos x$)
4. $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (Use the substitution $x = \sin u$)

Key Steps for Substitution:

- Let $u = g(x)$, then $\frac{du}{dx} = g'(x)$ or $du = g'(x)\,dx$.
- Rewrite the integral in terms of u and du.
- Integrate with respect to u and substitute back $g(x)$.

Problem 5: Mixed Applications

Problem Statement

Solve the following problems involving mixed integration techniques:

1. $\int (x^2+1)e^{x^3+3x}\,dx$
2. $\int \frac{\ln x}{x}\,dx$
3. $\int \sin(3x)\cdot e^{\cos(3x)}\,dx$
4. $\int \frac{1}{(x^2+4)^2}\cdot 2x\,dx$

Key Concepts and Formulas

Important Relationships

1. **Basic Integration Rules**:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \sin x\,dx = -\cos x + C, \quad \int \cos x\,dx = \sin x + C$$

$$\int e^x\,dx = e^x + C, \quad \int \frac{1}{x}\,dx = \ln|x| + C$$

2. **Integration of Composites**:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C, \quad \text{where } F'(x) = f(x)$$

3. **Reverse Chain Rule**:

$$\int g'(x)f(g(x))\,dx = F(g(x)) + C, \quad \text{where } F'(x) = f(x)$$

4. **Substitution**:

$$\text{Let } u = g(x), \quad \text{then } du = g'(x)\,dx$$

Rewrite the integral in terms of u and du, integrate, and substitute back $g(x)$.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct application of basic integration rules [2 marks per integral]
- Proper use of the constant of integration C [1 mark per integral]

Problem 2:

- Correct handling of linear composites [3 marks per integral]
- Proper division by the coefficient of x [1 mark per integral]

Problem 3:

- Correct identification of the inner function $g(x)$ [2 marks per integral]
- Correct application of the reverse chain rule [3 marks per integral]

Problem 4:

- Correct substitution and rewriting of the integral [2 marks per integral]
- Correct integration with respect to u [2 marks per integral]
- Proper substitution back to x [1 mark per integral]

Problem 5:

- Correct identification of the technique required (basic, composite, reverse chain rule, or substitution) [2 marks per integral]
- Correct integration and final answer [3 marks per integral]

11 Evaluating Definite Integrals

Problem 1: Evaluating Definite Integrals Analytically

Problem Statement

Evaluate the following definite integrals analytically:

1. $\int_1^3 (2x^2 - 3x + 1)\,dx$
2. $\int_0^\pi (\sin x - \cos x)\,dx$
3. $\int_1^e \frac{1}{x}\,dx$

Key Formula:

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

Problem 2: Areas Enclosed by a Curve and the x-Axis

Problem Statement

Find the area of the region enclosed by the curve $y = x^3 - 4x^2 + 3x$ and the x-axis between $x = 0$ and $x = 3$.

1. Identify the intervals where $y = x^3 - 4x^2 + 3x$ is positive and negative.
2. Split the integral into separate parts for positive and negative regions.
3. Evaluate the total area using:

$$A = \int_a^b |y|\,dx$$

Problem 3: Areas Between Two Curves

Problem Statement

Find the area of the region enclosed between the curves $y_1 = x^2$ and $y_2 = 2x + 3$ for $x \in [0, 3]$.

1. Find the points of intersection of the two curves.
2. Set up the integral for the area between the curves:

$$A = \int_a^b (y_2 - y_1)\, dx$$

3. Evaluate the integral to find the total area.

Problem 4: Using Technology for Definite Integrals

Problem Statement

Use a graphing calculator or software to evaluate the following definite integrals:

1. $\int_0^5 e^{x^2}\, dx$
2. $\int_{-1}^1 \sqrt{1 - x^2}\, dx$
3. $\int_0^\infty \frac{1}{1+x^2}\, dx$

Note: Some of these integrals cannot be solved analytically and require numerical methods or technology.

Problem 5: Mixed Applications

Problem Statement

Solve the following problems involving definite integrals and areas:

1. Find the total area enclosed by the curve $y = \sin x$ and the x-axis for $x \in [0, 2\pi]$.
2. Find the area of the region enclosed between the curves $y_1 = x^3$ and $y_2 = x^2$ for $x \in [0, 1]$.
3. Use technology to evaluate $\int_0^1 e^{-x^2}\, dx$ and interpret the result.

Key Concepts and Formulas

Important Relationships

1. **Definite Integral**:

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

2. **Area Enclosed by a Curve**:

$$A = \int_a^b |y|\,dx$$

Split the integral into positive and negative parts if y changes sign.

3. **Area Between Two Curves**:

$$A = \int_a^b (y_2 - y_1)\,dx, \quad \text{where } y_2 \geq y_1 \text{ on } [a, b]$$

4. **Using Technology**: Some integrals, such as $\int e^{x^2}\,dx$, cannot be solved analytically and require numerical methods or graphing tools.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct evaluation of definite integrals [3 marks per integral]
- Proper use of limits of integration [1 mark per integral]

Problem 2:

- Correct identification of positive and negative regions [2 marks]
- Correct splitting of the integral [2 marks]
- Correct evaluation of the total area [3 marks]

Problem 3:

- Correct points of intersection [2 marks]
- Correct setup of the integral [2 marks]
- Correct evaluation of the area [3 marks]

Problem 4:

- Correct use of technology to evaluate integrals [3 marks per integral]
- Proper interpretation of results [1 mark per integral]

Problem 5:

- Correct setup and evaluation of integrals [3 marks per part]
- Proper interpretation of results [2 marks per part]

Higher Level

12 Continuity and Differentiability

Problem 1: Continuity of a Function at a Point

Problem Statement

Determine whether the following functions are continuous at the given points:

1. $f(x) = \begin{cases} x^2 - 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ at $x = 1$.

2. $g(x) = \begin{cases} \sin x & \text{if } x \leq \frac{\pi}{2} \\ 1 & \text{if } x > \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$.

3. $h(x) = \frac{x^2-4}{x-2}$ at $x = 2$.

Key Steps:

- A function $f(x)$ is continuous at $x = a$ if:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Problem 2: Differentiability of a Function at a Point

Problem Statement

Determine whether the following functions are differentiable at the given points:

1. $f(x) = |x|$ at $x = 0$.

2. $g(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$ at $x = 1$.

3. $h(x) = \sqrt{|x|}$ at $x = 0$.

Key Steps:

- A function $f(x)$ is differentiable at $x = a$ if:

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Problem 3: Evaluating Limits

Problem Statement

Evaluate the following limits or determine whether the function diverges to infinity:

1. $\lim_{x\to 2} \frac{x^2-4}{x-2}$
2. $\lim_{x\to\infty} \frac{3x^2+5x}{2x^2-x+1}$
3. $\lim_{x\to 0^+} \ln x$
4. $\lim_{x\to\infty} e^{-x}$

Key Concepts:

- Use algebraic simplification, L'Hôpital's Rule, or known limits to evaluate.
- A function diverges to infinity if $\lim_{x\to a} f(x) = \infty$ or $-\infty$.

Problem 4: Derivatives from First Principles

Problem Statement

Find the derivative of the following functions using first principles:

1. $f(x) = x^2$
2. $g(x) = 3x + 5$
3. $h(x) = \frac{1}{x}$

Key Formula:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Problem 5: Higher Derivatives

Problem Statement

Find the first, second, and third derivatives of the following functions:

1. $f(x) = x^4 - 3x^3 + 2x^2 - x + 7$
2. $g(x) = e^x$
3. $h(x) = \sin x$

Key Notation:

- First derivative: $f'(x)$ or $\frac{dy}{dx}$
- Second derivative: $f''(x)$ or $\frac{d^2y}{dx^2}$
- Third derivative: $f'''(x)$ or $\frac{d^3y}{dx^3}$
- Higher derivatives: $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$

Key Concepts and Formulas

Important Relationships

1. **Continuity**:

   A function $f(x)$ is continuous at $x = a$ if: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

2. **Differentiability**:

   A function $f(x)$ is differentiable at $x = a$ if: $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$.

3. **Limits**:

   $\lim_{x \to a} f(x)$ can be evaluated using algebraic simplification, L'Hôpital's Rule, or known limits.

4. **First Principles**:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

5. **Higher Derivatives**:

$$f^{(n)}(x) = \frac{d^n y}{dx^n}, \quad \text{where } n \text{ is the order of the derivative.}$$

Marking Guidelines

Marking Scheme

Problem 1:

- Correct evaluation of left-hand and right-hand limits [2 marks per function]
- Correct conclusion about continuity [1 mark per function]

Problem 2:

- Correct evaluation of left-hand and right-hand derivatives [2 marks per function]
- Correct conclusion about differentiability [1 mark per function]

Problem 3:

- Correct simplification or application of L'Hôpital's Rule [2 marks per limit]
- Correct conclusion about convergence or divergence [1 mark per limit]

Problem 4:

- Correct application of first principles [3 marks per function]
- Correct final derivative [2 marks per function]

Problem 5:

- Correct first, second, and third derivatives [2 marks per derivative]
- Correct use of higher derivative notation [1 mark per function]

13 L'Hôpital's Rule and Evaluation of Limits

Problem 1: Basic Application of L'Hôpital's Rule

Problem Statement

Evaluate the following limits using L'Hôpital's Rule:

1. $\lim_{x\to 0} \frac{\sin x}{x}$
2. $\lim_{x\to\infty} \frac{e^x}{x^2}$
3. $\lim_{x\to 1} \frac{\ln x}{x-1}$

Key Formula:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \quad \text{if } \lim_{x\to a} \frac{f(x)}{g(x)} \text{ is of the form } \frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Problem 2: Repeated Use of L'Hôpital's Rule

Problem Statement

Evaluate the following limits using repeated applications of L'Hôpital's Rule:

1. $\lim_{x\to\infty} \frac{x^2}{e^x}$
2. $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$
3. $\lim_{x\to\infty} \frac{\ln x}{x}$

Problem 3: Limits Using the Maclaurin Series

Problem Statement

Evaluate the following limits by replacing the numerator and denominator with their Maclaurin series:

1. $\lim_{x\to 0} \frac{\sin x}{x}$
2. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$
3. $\lim_{x\to 0} \frac{e^x-1}{x}$

Key Formula:

- Maclaurin series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Problem 4: Mixed Applications of L'Hôpital's Rule and Maclaurin Series

Problem Statement

Solve the following problems using either L'Hôpital's Rule or the Maclaurin series:

1. $\lim_{x\to 0} \frac{\ln(1+x)}{x}$
2. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$
3. $\lim_{x\to \infty} \frac{x}{e^x}$
4. $\lim_{x\to 0} \frac{e^{x^2}-1}{x^2}$

Key Concepts and Formulas

Important Relationships

1. **L'Hôpital's Rule**:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad \text{if } \lim_{x \to a} \frac{f(x)}{g(x)} \text{ is of the form } \frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

2. **Maclaurin Series**:
   - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
   - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
   - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
   - $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad |x| < 1$
3. **Repeated Use of L'Hôpital's Rule**:

   Apply L'Hôpital's Rule repeatedly until the limit can be evaluated.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct application of L'Hôpital's Rule [2 marks per limit]
- Correct evaluation of the limit [1 mark per limit]

Problem 2:

- Correct repeated application of L'Hôpital's Rule [3 marks per limit]
- Correct evaluation of the limit [2 marks per limit]

Problem 3:

- Correct substitution of the Maclaurin series [3 marks per limit]
- Correct simplification and evaluation of the limit [2 marks per limit]

Problem 4:

- Correct identification of the appropriate method (L'Hôpital's Rule or Maclaurin series) [2 marks per limit]
- Correct application of the method and evaluation of the limit [3 marks per limit]

14 Applications of Differentiation

Problem 1: Implicit Differentiation

Problem Statement

Find $\frac{dy}{dx}$ for the following equations using implicit differentiation:

1. $x^2 + y^2 = 25$
2. $x^3 + y^3 = 6xy$
3. $\sin(xy) = x + y$

Key Steps:

- Differentiate both sides of the equation with respect to x, treating y as a function of x.
- Apply the chain rule to terms involving y.
- Solve for $\frac{dy}{dx}$.

Problem 2: Related Rates of Change

Problem Statement

Solve the following related rates problems:

1. A spherical balloon is being inflated so that its volume increases at a rate of $100\,\text{cm}^3/\text{s}$. How fast is the radius increasing when the radius is $5\,\text{cm}$?
Hint: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.
2. A ladder $10\,\text{m}$ long is leaning against a vertical wall. The bottom of the ladder is pulled away from the wall at a rate of $1\,\text{m/s}$. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is $6\,\text{m}$ from the wall?
Hint: Use the Pythagorean theorem: $x^2 + y^2 = 10^2$.
3. Water is being poured into a conical tank at a rate of $50\,\text{cm}^3/\text{s}$. The tank has a height of $12\,\text{cm}$ and a base radius of $6\,\text{cm}$. How fast is the water level rising when the water is $4\,\text{cm}$ deep?
Hint: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, and the radius and height are proportional.

Problem 3: Optimization Problems

Problem Statement

Solve the following optimization problems:

1. A farmer wants to build a rectangular pen with a fixed perimeter of $100\,\text{m}$. What dimensions will maximize the area of the pen? What is the maximum area?

2. A box with a square base and no top is to be made from $48\,\text{m}^2$ of material. What dimensions will maximize the volume of the box?

3. A company wants to design a cylindrical can that holds $500\,\text{cm}^3$ of liquid. What dimensions (radius and height) will minimize the surface area of the can?
   Hint: The volume of a cylinder is $V = \pi r^2 h$, and the surface area is $A = 2\pi r^2 + 2\pi rh$.

Key Steps:

- Write the function to be optimized in terms of a single variable.
- Find the critical points by solving $f'(x) = 0$.
- Use the second derivative test or analyze the endpoints to confirm whether the critical points are maxima or minima.

Key Concepts and Formulas

Important Relationships

1. **Implicit Differentiation**:

   Differentiate both sides of the equation with respect to x, treating y as a function of x.

2. **Related Rates**:

   Use the chain rule to relate the rates of change of different variables.

   Example: If $V = \frac{4}{3}\pi r^3$, then:

   $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

3. **Optimization**:

   - Write the function to be optimized in terms of a single variable.
   - Find critical points by solving $f'(x) = 0$.
   - Use the second derivative test or endpoints to confirm maxima or minima.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct differentiation of both sides of the equation [2 marks per part]
- Correct application of the chain rule [2 marks per part]
- Correct solution for $\frac{dy}{dx}$ [1 mark per part]

Problem 2:

- Correct identification of the relationship between variables [2 marks per part]
- Correct differentiation with respect to time [2 marks per part]
- Correct solution for the rate of change [2 marks per part]

Problem 3:

- Correct formulation of the function to be optimized [2 marks per part]
- Correct critical points and verification of maxima/minima [3 marks per part]
- Correct final solution with interpretation [2 marks per part]

15 Further Derivatives and Integrals

Problem 1: Derivatives of Trigonometric, Exponential, and Logarithmic Functions

Problem Statement

Find the derivatives of the following functions:

1. $f(x) = \tan(3x)$
2. $g(x) = \sec^2(2x + 1)$
3. $h(x) = \arcsin(x^2)$
4. $k(x) = \log_2(x^3)$
5. $m(x) = e^{2x} \cdot \arctan(x)$

Key Formulae:

- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Problem 2: Indefinite Integrals of Standard Functions

Problem Statement

Evaluate the following indefinite integrals:

1. $\int \sec^2(3x)\,dx$
2. $\int \frac{1}{\sqrt{1-x^2}}\,dx$
3. $\int \frac{1}{1+x^2}\,dx$
4. $\int e^{2x}\,dx$
5. $\int \frac{1}{x\ln 2}\,dx$

Key Formulae:

- $\int \sec^2 x\,dx = \tan x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$

Problem 3: Integration of Composites with Linear Functions

Problem Statement

Evaluate the following integrals:

1. $\int \sec^2(2x+1)\,dx$
2. $\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$
3. $\int e^{5x+2}\,dx$
4. $\int \frac{1}{(2x+3)\ln 5}\,dx$

Key Formula:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C, \quad \text{where } F'(x) = f(x).$$

Problem 4: Integration Using Partial Fractions

Problem Statement

Evaluate the following integrals by first splitting the integrand into partial fractions:

1. $\int \frac{1}{x^2-1}\,dx$
2. $\int \frac{2x+3}{(x+1)(x^2+1)}\,dx$
3. $\int \frac{1}{(x+2)(x-1)}\,dx$
4. $\int \frac{x^2}{(x^2+1)(x-1)}\,dx$

Key Steps:

- Decompose the fraction into partial fractions:

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-r_1)} + \frac{B}{(x-r_2)} + \dots$$

- Integrate each term separately.

Key Concepts and Formulae

Important Relationships

1. **Derivatives of Standard Functions**:

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad \frac{d}{dx}(e^x) = e^x$$

2. **Indefinite Integrals of Standard Functions**:

$$\int \sec^2 x\, dx = \tan x + C, \quad \int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + C, \quad \int e^x\, dx = e^x + C$$

3. **Integration of Composites**:

$$\int f(ax+b)\, dx = \frac{1}{a}F(ax+b) + C, \quad \text{where } F'(x) = f(x)$$

4. **Partial Fractions**:

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-r_1)} + \frac{B}{(x-r_2)} + \dots$$

Decompose the fraction and integrate each term separately.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct differentiation of each function [2 marks per part]
- Correct application of chain, product, or quotient rules [2 marks per part]

Problem 2:

- Correct identification of the standard integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]

Problem 3:

- Correct handling of the composite function [2 marks per part]
- Correct division by the coefficient of x [1 mark per part]
- Correct evaluation of the integral [2 marks per part]

Problem 4:

- Correct decomposition into partial fractions [3 marks per part]
- Correct integration of each term [2 marks per part]

16 Advanced Integration Techniques

Problem 1: Integration by Substitution

Problem Statement

Evaluate the following integrals using the given substitution:

1. $\int x\sqrt{x^2+1}\,dx$, with the substitution $u = x^2 + 1$.
2. $\int \frac{\sin(2x)}{\cos^2(2x)}\,dx$, with the substitution $u = \cos(2x)$.
3. $\int_0^1 xe^{x^2}\,dx$, with the substitution $u = x^2$.
4. $\int_0^{\pi/2} \sin^2 x\,dx$, with the substitution $u = \cos x$.

Key Steps:

- Let $u = g(x)$, then $du = g'(x)\,dx$.
- Rewrite the integral in terms of u and du.
- For definite integrals, change the limits of integration:

$$\text{New limits: } u_1 = g(a),\ u_2 = g(b).$$

Problem 2: Integration by Parts

Problem Statement

Evaluate the following integrals using integration by parts:

1. $\int xe^x \, dx$
2. $\int x \ln x \, dx$
3. $\int e^x \sin x \, dx$
4. $\int_0^1 x^2 e^x \, dx$

Key Formula:

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

Choosing u and v:

- Use the **LIATE** rule to choose u:
  - Logarithmic functions ($\ln x$)
  - Inverse trigonometric functions ($\arcsin x$, $\arctan x$)
  - Algebraic functions (x, x^2)
  - Trigonometric functions ($\sin x$, $\cos x$)
  - Exponential functions (e^x)

Problem 3: Repeated Integration by Parts

Problem Statement

Evaluate the following integrals using repeated integration by parts:

1. $\int x^2 e^x \, dx$
2. $\int x^2 \ln x \, dx$
3. $\int e^x \cos x \, dx$
4. $\int_0^{\pi/2} x^2 \sin x \, dx$

Key Steps:

- Apply the integration by parts formula twice.
- For integrals like $\int e^x \cos x \, dx$, use the result of the second application to form an equation and solve for the integral.

Problem 4: Mixed Applications of Substitution and Integration by Parts

Problem Statement

Solve the following integrals using a combination of substitution and integration by parts:

1. $\int x^2\sqrt{x^2+1}\,dx$
2. $\int xe^{x^2}\,dx$
3. $\int_0^1 x^2\ln(1+x^2)\,dx$
4. $\int_0^{\pi/4}\tan x\ln(\cos x)\,dx$

Key Concepts and Formulae

Important Relationships

1. **Integration by Substitution**:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \quad \text{where } u = g(x).$$

For definite integrals:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

2. **Integration by Parts**:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

3. **Repeated Integration by Parts**: For integrals like $\int e^x \cos x\,dx$, repeated application of integration by parts leads to:

$$\int e^x \cos x\,dx = e^x(\sin x + \cos x)/2 + C$$

4. **Choosing u and v**: Use the **LIATE** rule to choose u:

- Logarithmic functions ($\ln x$)
- Inverse trigonometric functions ($\arcsin x$, $\arctan x$)
- Algebraic functions (x, x^2)
- Trigonometric functions ($\sin x$, $\cos x$)
- Exponential functions (e^x)

Marking Guidelines

Marking Scheme

Problem 1:

- Correct substitution and rewriting of the integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]
- Correct change of limits for definite integrals [1 mark per part]

Problem 2:

- Correct choice of u and v [2 marks per part]
- Correct application of the integration by parts formula [3 marks per part]
- Correct evaluation of the integral [2 marks per part]

Problem 3:

- Correct repeated application of integration by parts [3 marks per part]
- Correct handling of resulting equations (if applicable) [2 marks per part]
- Correct final evaluation of the integral [2 marks per part]

Problem 4:

- Correct identification of the appropriate method (substitution or integration by parts) [2 marks per part]
- Correct application of the method [3 marks per part]
- Correct final evaluation of the integral [2 marks per part]

17 Further Areas and Volumes

Problem 1: Area of the Region Enclosed by a Curve and the y-Axis

Problem Statement

Find the area of the region enclosed by the curve and the y-axis for the following:

1. $x = \sqrt{y}$, for $0 \leq y \leq 4$.
2. $x = y^2 - 2y$, for $-1 \leq y \leq 2$.
3. $x = \ln(y)$, for $1 \leq y \leq e$.

Key Formula:

$$\text{Area} = \int_c^d |x|\, dy$$

Steps:

- Express x in terms of y if necessary.
- Evaluate the integral of $|x|$ over the given limits.

Problem 2: Volume of Revolution About the x-Axis

Problem Statement

Find the volume of the solid obtained by rotating the region enclosed by the curve about the x-axis:

1. $y = x^2$, for $0 \leq x \leq 2$.
2. $y = \sin x$, for $0 \leq x \leq \pi$.
3. $y = e^{-x}$, for $0 \leq x \leq 1$.

Key Formula:

$$\text{Volume} = \int_a^b \pi y^2\, dx$$

Steps:

- Square the function y to find y^2.
- Integrate πy^2 over the given limits.

Problem 3: Volume of Revolution About the y-Axis

Problem Statement

Find the volume of the solid obtained by rotating the region enclosed by the curve about the y-axis:

1. $x = \sqrt{y}$, for $0 \leq y \leq 4$.
2. $x = y^2$, for $0 \leq y \leq 1$.
3. $x = \ln(y)$, for $1 \leq y \leq e$.

Key Formula:

$$\text{Volume} = \int_c^d \pi x^2 \, dy$$

Steps:

- Express x in terms of y if necessary.
- Square the function x to find x^2.
- Integrate πx^2 over the given limits.

Problem 4: Mixed Applications of Areas and Volumes

Problem Statement

Solve the following problems involving areas and volumes:

1. Find the area of the region enclosed by $x = y^2$ and $x = 4 - y^2$.
2. Find the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = 4$ about the x-axis.
3. Find the volume of the solid obtained by rotating the region enclosed by $x = y^2$ and $x = 4$ about the y-axis.

Key Concepts and Formulae

Important Relationships

1. **Area Enclosed by a Curve and the y-Axis**:

$$\text{Area} = \int_c^d |x|\, dy$$

2. **Volume of Revolution About the x-Axis**:

$$\text{Volume} = \int_a^b \pi y^2\, dx$$

3. **Volume of Revolution About the y-Axis**:

$$\text{Volume} = \int_c^d \pi x^2\, dy$$

4. **Steps for Solving**:
   - Express the function in terms of the appropriate variable (x or y).
   - Square the function to find y^2 or x^2.
   - Integrate over the given limits.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct expression of x in terms of y (if necessary) [2 marks per part]
- Correct setup of the integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]

Problem 2:

- Correct expression of y^2 [2 marks per part]
- Correct setup of the integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]

Problem 3:

- Correct expression of x^2 [2 marks per part]
- Correct setup of the integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]

Problem 4:

- Correct identification of the region or volume [2 marks per part]
- Correct setup of the integral [2 marks per part]
- Correct evaluation of the integral [2 marks per part]

18 Differential Equations

Problem 1: Forming Differential Equations

Problem Statement

Form the differential equation for the following scenarios:

1. The rate of change of a population P is proportional to the population itself.
2. The rate of change of the temperature T of an object is proportional to the difference between its temperature and the surrounding temperature T_s.
3. The rate of change of a quantity y with respect to x is inversely proportional to the square of x.

Key Concept:

- The derivative $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
- Translate the given relationship into a differential equation.

Problem 2: Numerical Solution Using Euler's Method

Problem Statement

Use Euler's method to approximate the solution of the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$:

1. Use a step size of $h = 0.1$ to find $y(0.1)$ and $y(0.2)$.
2. Use a step size of $h = 0.2$ to find $y(0.2)$ and $y(0.4)$.

Key Formula:

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n)$$

Problem 3: Solving Differential Equations with Separable Variables

Problem Statement

Solve the following differential equations by separating variables:

1. $\frac{dy}{dx} = xy$
2. $\frac{dy}{dx} = \frac{x}{y}$
3. $\frac{dy}{dx} = \frac{y^2}{x^2}$

Key Steps:

- Rewrite the equation so that all terms involving y are on one side and all terms involving x are on the other.
- Integrate both sides.
- Use the initial condition (if provided) to find the constant of integration.

Problem 4: Homogeneous Differential Equations

Problem Statement

Solve the following homogeneous differential equations using the substitution $y = vx$:

1. $\frac{dy}{dx} = \frac{x+y}{x}$
2. $\frac{dy}{dx} = \frac{x^2+xy}{x^2}$
3. $\frac{dy}{dx} = \frac{y^2-x^2}{xy}$

Key Steps:

- Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
- Rewrite the equation in terms of v and x.
- Solve the resulting equation using separation of variables or another appropriate method.

Problem 5: Solving First-Order Linear Differential Equations

Problem Statement

Solve the following first-order linear differential equations using the integrating factor method:

1. $\frac{dy}{dx} + y = e^x$
2. $\frac{dy}{dx} - 2y = x$
3. $\frac{dy}{dx} + 3y = \sin x$

Key Formula:

- The general form of a first-order linear differential equation is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- The integrating factor is:

$$I = e^{\int P(x)\,dx}$$

- The solution is:

$$Iy = \int IQ(x)\,dx + C$$

Key Concepts and Formulae

Important Relationships

1. **Forming Differential Equations**:

   Translate the given rate of change into a differential equation.

2. **Euler's Method**:

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n)$$

3. **Separable Variables**:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

4. **Homogeneous Differential Equations**:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \quad \text{use } y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

5. **First-Order Linear Differential Equations**:

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \text{Integrating factor: } I = e^{\int P(x)\,dx}.$$

$$\text{Solution: } Iy = \int IQ(x)\,dx + C.$$

Marking Guidelines

Marking Scheme

Problem 1:

- Correct translation of the scenario into a differential equation [3 marks per part]

Problem 2:

- Correct application of Euler's method formula [2 marks per step]
- Correct numerical values for y_n [2 marks per step]

Problem 3:

- Correct separation of variables [2 marks per part]
- Correct integration of both sides [2 marks per part]
- Correct application of initial conditions (if provided) [1 mark per part]

Problem 4:

- Correct substitution $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ [2 marks per part]
- Correct simplification and solution of the resulting equation [3 marks per part]

Problem 5:

- Correct identification of $P(x)$ and $Q(x)$ [1 mark per part]
- Correct calculation of the integrating factor [2 marks per part]
- Correct solution of the equation [3 marks per part]

19 Maclaurin Series

Problem 1: Deriving Basic Maclaurin Series

Problem Statement

Derive the Maclaurin series for the following functions up to the x^4 term:

1. e^x
2. $\sin x$
3. $\cos x$
4. $\ln(1+x)$
5. $(1+x)^p$, where $p \in \mathbb{R}$

Key Formula:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

Problem 2: Substitution, Products, Integration, and Differentiation of Maclaurin Series

Problem Statement

Use substitution, products, integration, or differentiation to find the Maclaurin series for the following functions up to the x^4 term:

1. e^{x^2}
2. $\sin(2x)$
3. $\int_0^x e^t \, dt$
4. $\frac{\sin x}{x}$
5. $\ln(1+x^2)$

Key Steps:

- Substitute x^2, $2x$, or other expressions into the basic Maclaurin series.
- Multiply or divide two series term-by-term.
- Integrate or differentiate the series term-by-term.

Problem 3: Multiplying Two Maclaurin Series

Problem Statement

Find the Maclaurin series for the following products up to the x^4 term:

1. $e^x \cos x$
2. $\sin x \ln(1+x)$
3. $(1+x)^2 \cdot e^x$

Key Steps:

- Write the Maclaurin series for each function.
- Multiply the series term-by-term, keeping terms up to x^4.

Problem 4: Maclaurin Series from Differential Equations

Problem Statement

Find the Maclaurin series for the solution of the following differential equations:

1. $\frac{dy}{dx} = y$, with $y(0) = 1$.
2. $\frac{d^2y}{dx^2} + y = 0$, with $y(0) = 1$ and $y'(0) = 0$.
3. $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$, with $y(0) = 1$ and $y'(0) = 1$.

Key Steps:

- Assume $y = \sum_{k=0}^{\infty} a_k x^k$.
- Substitute y, y', and y'' into the differential equation.
- Compare coefficients of x^k to find a recurrence relation for a_k.
- Use the initial conditions to find the first few terms of the series.

Problem 5: Differentiating and Integrating Maclaurin Series

Problem Statement

Differentiate or integrate the following Maclaurin series term-by-term to find the series for the given functions up to the x^4 term:

1. $\frac{d}{dx}(\ln(1+x))$
2. $\int e^x\,dx$
3. $\frac{d}{dx}(\sin x \cdot e^x)$
4. $\int \cos x\,dx$

Key Steps:

- Differentiate or integrate each term of the series.
- Keep terms up to x^4.

Key Concepts and Formulae

Important Relationships

1. **Maclaurin Series Formula**:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

2. **Basic Maclaurin Series**:
   - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
   - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
   - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
   - $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad |x| < 1$
   - $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$
3. **Substitution**: Replace x with $g(x)$ in the series.
4. **Products**: Multiply two series term-by-term.
5. **Integration and Differentiation**: Integrate or differentiate each term of the series.
6. **Differential Equations**: Substitute $y = \sum_{k=0}^{\infty} a_k x^k$ into the equation and compare coefficients.

Marking Guidelines

Marking Scheme

Problem 1:

- Correct derivatives at $x = 0$ [2 marks per function]
- Correct series up to x^4 [2 marks per function]

Problem 2:

- Correct substitution or manipulation of the series [2 marks per function]
- Correct series up to x^4 [2 marks per function]

Problem 3:

- Correct multiplication of series [3 marks per function]
- Correct series up to x^4 [2 marks per function]

Problem 4:

- Correct substitution of y, y', and y'' into the differential equation [3 marks per equation]
- Correct recurrence relation for a_k [2 marks per equation]
- Correct series up to x^4 [2 marks per equation]

Problem 5:

- Correct differentiation or integration of each term [2 marks per function]
- Correct series up to x^4 [2 marks per function]

Conclusion

Mathematics is not just about understanding theory; it is about applying concepts to solve problems effectively. This guide has provided you with a collection of expertly crafted practice problems focused on calculus, designed to challenge your understanding and enhance your problem-solving skills.

For detailed solutions and answers, keep following me — they will be available soon! If you're looking for personalized guidance, book a one-on-one mentorship session with me to deepen your understanding of IB Mathematics AA/AI HL, calculus, or even Olympiad-level problems. Together, we can build the confidence and skills you need to excel in mathematics.

As you prepare for your exams, remember:

- **Practice is the key to success**: The more problems you solve, the more confident and efficient you become. Focus on understanding the logic behind each solution rather than memorizing formulas.
- **Learn from mistakes**: Every mistake is an opportunity to grow. Analyze where you went wrong and refine your approach.
- **Time management is crucial**: Simulate exam conditions to improve your speed and accuracy under pressure.

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- Rishabh Kumar
Founder, Mathematics Elevate Academy
Elite Mentor for IB Mathematics
Alumnus of IIT Guwahati & Indian Statistical Institute

Thank You!

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