International Baccalaureate (IB) Diploma Programme Mathematics Analysis and Approaches Higher Level

Paper 3 Series - Part 1

The IB 7-Scorer's Ultimate Guide

Crafted Exclusively for High-Achieving IB Mathematics Students: March 2025 Edition

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Introduction

The IB 7-Scorer's Ultimate Guide — March 2025 Edition is meticulously crafted for IB DP Mathematics students aiming to excel in Analysis & Approaches HL Paper 3, following the new syllabus (announced in 2019) for exams from 2021 onwards. This premier guide offers expertly curated high-level problems, advanced challenges, and multiple solution pathways, empowering students to master complex concepts with confidence. Dive into examiner-style solutions, marking scheme breakdowns, and insightful commentary on common pitfalls. Elevate your understanding with problem-solving strategies that go beyond the IB syllabus, offering a true test of mathematical prowess. Each solution is presented with step-by-step clarity, expert insights, and refined techniques, ensuring a comprehensive and enriching learning experience.

Need personalized guidance? Book a one-on-one mentorship session with me and get tailored support to confidently conquer IB Math AA.

Problem 1 : [27 Points]

Problem Description

This problem investigates two methods for triggering random events in a game. Each action has a random chance to be boosted, and the game developer is considering two potential models to determine when these boosts occur.

For (a) to (d)

In the first model, the probability of an action being boosted is fixed for every attempt.

Problem 1 (a)(i)

Problem (a) (i)

Suppose the probability of an action being boosted is 0.2.

(i) Calculate the probability that the first boost happens on the third action.

[2 Points]

Solutions

Solution

Step 1: Identifying the probability distribution: *Geometric distribution*.

Step 2: Applying the geometric probability formula:

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

Step 3: Substituting the values and computing:

$$P(X = 3) = (0.8)^2 \times 0.2 = 0.128.$$

Final Answer: The probability is **0.128**.

Another Solution

Step 1: Understanding the scenario: The first two actions must not be boosted, and the third action must be boosted.

Step 2: Calculating individual probabilities:
- Probability of no boost on the first action $= 0.8$
- Probability of no boost on the second action $= 0.8$
- Probability of boost on the third action $= 0.2$

Step 3: Multiplying the probabilities:

$$P = 0.8 \times 0.8 \times 0.2 = 0.128$$

Final Answer: The probability is **0.128**.

Examiner's Expectation

The examiner expects students to correctly identify the probability distribution and apply the formula accurately. Marks are awarded as follows:

- **Method (M1):** Correctly setting up the probability calculation.
- **Accuracy (A1):** Correctly computing the final probability.

[2 Points]

Critical Points

- **Key Formula:** The geometric probability formula is used to calculate the probability of the first success occurring on the k-th trial:

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

- **Complement Rule:** The complement rule is useful for calculating probabilities such as "at least one success":

$$P(\text{At Least One}) = 1 - P(\text{None})$$

- **Common Mistake:** Students often forget to subtract from 1 when using the complement method, leading to incorrect results.

This section explores advanced concepts related to probability distributions, helping deepen your understanding beyond the basic problem.

Definition
The geometric distribution models the number of trials until the first success in a sequence of independent Bernoulli trials.
Key Requirements

- Binary outcomes: Each trial has exactly two possible outcomes
- Constant probability: Success probability p remains unchanged
- Independence: Trials do not affect each other

Probability Mass Function (PMF)

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

Application Example
In our problem, this distribution models the number of attempts until the first boost occurs.

A. Multiplication Law

- For independent events: $P(A \cap B) = P(A) \times P(B)$
- For dependent events: $P(A \cap B) = P(A) \times P(B|A)$

Example: Two coin flips

$$P(\text{Two Heads}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

B. Addition Law

- For mutually exclusive events: $P(A \cup B) = P(A) + P(B)$
- For non-mutually exclusive events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: Rolling a die

$$P(\text{3 or 5}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

A. Binomial Distribution

- Models number of successes in fixed trials
- PMF: $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$
- Key parameters: n (trials), p (probability)

B. Hypergeometric Distribution *(Beyond IB Syllabus)*

- Sampling without replacement
- Population size is finite
- Probability changes after each draw

Consider:

1. How does the geometric distribution relate to the exponential distribution?
2. When is it appropriate to use each probability distribution?
3. What happens to probabilities in very large numbers of trials?

Key Takeaways

- Geometric distribution models time until first success
- Multiplication law for "and" scenarios
- Addition law for "or" scenarios
- Different distributions for different sampling methods

The probability mass function is given by:

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Parameters:

- N: Total population size
- K: Number of successes in population
- n: Sample size (number of draws)
- k: Observed successes

Example Application: Consider drawing cards from a standard deck:

- Total cards: $N = 52$
- Red cards: $K = 26$
- Cards drawn: $n = 5$
- Red cards wanted: $k = 2$

Definition: Models the number of events occurring in a fixed interval, assuming:

- Events occur independently
- Events occur at a constant average rate
- Only one event can occur at any instant

Probability Mass Function:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

where:

- λ: Average rate of events
- k: Number of events observed
- e: Euler's number (≈ 2.71828)

Practical Example: Call center receiving average 5 calls/hour:

$$P(X = 3) = \frac{e^{-5} \cdot 5^3}{3!} \approx 0.140$$

Essential Characteristics:

- Countable outcomes
- Each outcome has a probability
- Total probability $= 1$

Key Distributions in IB Syllabus:

1. **Binomial Distribution**
   - Models successes in fixed trials
   - PMF: $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$
2. **Poisson Distribution**
   - Models events in fixed interval
   - PMF: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

Expected Value:

$$E(X) = \sum x \cdot P(X = x)$$

Variance:

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \sum (x - E(X))^2 \cdot P(X = x)$$

Properties:

- All probabilities are non-negative
- Sum of all probabilities equals 1
- Variance is always non-negative

Practice Example

Problem: Fair coin flipped 5 times

- Trials: $n = 5$
- Success probability: $p = 0.5$
- Find: $P(X = 3)$ (exactly 3 heads)

Solution:

$$P(X = 3) = \binom{5}{3}(0.5)^3(0.5)^2 = 0.3125$$

Key Takeaways

- Different distributions model different scenarios
- Choose distribution based on problem context
- Focus on Binomial and Poisson for IB syllabus
- Practice identifying appropriate distributions

Quick Revision

- **Key Formula:** The probability of the first success occurring on the k-th trial in a geometric distribution is:

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

  where:

  - p: Probability of success in a single trial
  - k: Trial number of the first success

- **Complement Rule:** The probability of at least one success in n trials is:

$$P(\text{At Least One}) = 1 - P(\text{None})$$

  where:

$$P(\text{None}) = (1 - p)^n$$

- **Common Mistake:** Forgetting to subtract from 1 when using the complement method, leading to incorrect results. Always verify whether the problem asks for "at least one" or "none."

Try Solving These!

1. Find the probability that the first boost occurs on the **fifth action**.
2. Calculate the probability of **exactly two boosts** in the first **eight actions**.
3. What is the probability of **at least one boost** in the first **ten actions**?
4. If the probability of a boost is $p = 0.3$, how many actions are expected before the first boost occurs? (Hint: Use the expected value formula for the geometric distribution.)

Note: Use the following formulas to solve:

- **Geometric Distribution:** $P(X = k) = (1 - p)^{k-1} \cdot p$
- **Binomial Distribution:** $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$
- **Complement Rule:** $P(\text{At Least One}) = 1 - P(\text{None})$

Try Solving These!

1. Find the probability that **at most three boosts** occur by the **sixth action**.
2. Calculate the probability of **exactly two boosts** in the first **eight actions**.

Problem 1 (a)(ii)

Problem (a) (ii)

Suppose the probability of an action being boosted is 0.2.

(ii) Find the probability that the player receives **at least one boost** within the first **six actions**. **[3 Points]**

Solutions

Solution: Complementary Method

Step 1: Calculate the probability of **no boosts** in six actions.
The probability of no boost in a single action is $1-p=0.8$. For six actions:

$$P(\text{No Boost}) = (1-p)^6 = (0.8)^6 \approx 0.262144.$$

Step 2: Use the complement rule to find the probability of **at least one boost**:

$$P(\text{At Least One Boost}) = 1 - P(\text{No Boost}) = 1 - 0.262144 = 0.737856.$$

Final Answer: The probability is **0.738**.

Alternative Solution: Direct Calculation (Not Recommended for Large n)

Step 1: Calculate the probability of receiving at least one boost directly. This involves summing the probabilities of receiving exactly 1, 2, 3, ..., or 6 boosts. For example:

$$P(\text{At Least One Boost}) = P(X = 1) + P(X = 2) + \cdots + P(X = 6).$$

Step 2: Use the binomial probability formula for each term:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

Step 3: Compute each term and add them. However, this method is computationally intensive and less efficient than the complement rule.
Note: The complement method is preferred for larger n.

Examiner's Expectation

Method (M1): Correctly identify and apply the complement rule:

$$P(\text{At Least One Boost}) = 1 - P(\text{No Boost}).$$

Accuracy (A1): Correctly compute:

$$P(\text{No Boost}) = (0.8)^6 \quad \text{and} \quad P(\text{At Least One Boost}) = 1-0.262144 = 0.738.$$

[3 Points]

Critical Points

- **Key Formula:** For geometric or binomial distributions:

$$P(X = k) = (1 - p)^{k-1} \cdot p \quad \text{(Geometric Distribution)}$$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{(Binomial Distribution)}.$$

- **Complement Rule:** For "at least one" scenarios:

$$P(\text{At Least One}) = 1 - P(\text{None}).$$

- **Common Mistake:** Forgetting to subtract from 1 when using the complement method, or incorrectly calculating $P(\text{No Boost})$.

Exploration Beyond This Problem

Exploring beyond this problem helps strengthen your understanding of probability distributions and their applications. Let's dive deeper into related concepts and scenarios!

1. Geometric Distribution vs. Binomial Distribution

Geometric Distribution:

- Models the number of trials until the first success.
- Probability mass function (PMF):

$$P(X = k) = (1 - p)^{k-1} \cdot p$$

- Example: Finding the probability that the first boost occurs on the 5th action.

Binomial Distribution:

- Models the number of successes in a fixed number of trials.
- Probability mass function (PMF):

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

- Example: Calculating the probability of exactly 2 boosts in the first 8 actions.

Key Difference: The geometric distribution focuses on the trial of the first success, while the binomial distribution focuses on the total number of successes in a fixed number of trials.

2. Poisson Distribution: Modeling Rare Events

The **Poisson Distribution** is used to model the number of events occurring in a fixed interval of time or space, assuming:

- Events occur independently.
- Events occur at a constant average rate (λ).

Probability Mass Function (PMF):

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Example: Suppose a game developer observes that, on average, 3 boosts occur every 10 minutes. What is the probability of exactly 2 boosts occurring in the next 10 minutes? Use $\lambda = 3$ and $k = 2$:

$$P(X = 2) = \frac{e^{-3} \cdot 3^2}{2!} \approx 0.224.$$

3. Hypergeometric Distribution: Sampling Without Replacement

The **Hypergeometric Distribution** models the probability of k successes in n draws from a population of size N containing K successes, without replacement.
Probability Mass Function (PMF):

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Example: A deck of 52 cards contains 26 red cards. If you draw 5 cards, what is the probability of getting exactly 2 red cards? Use $N = 52$, $K = 26$, $n = 5$, and $k = 2$:

$$P(X = 2) = \frac{\binom{26}{2}\binom{26}{3}}{\binom{52}{5}} \approx 0.325.$$

4. Thought-Provoking Questions

1. How does the geometric distribution relate to the exponential distribution in continuous probability?
2. When should you use the Poisson distribution instead of the binomial distribution?
3. What happens to the binomial distribution as the number of trials (n) becomes very large and the probability of success (p) becomes very small?
4. How does the hypergeometric distribution differ from the binomial distribution?

Key Takeaways

- The geometric distribution models the trial of the first success, while the binomial distribution models the total number of successes in fixed trials.
- The Poisson distribution is ideal for modeling rare events in a fixed interval.
- The hypergeometric distribution is used for sampling without replacement.
- Understanding the differences between these distributions helps in selecting the right model for real-world problems.

Quick Revision

- **Key Formula:**
$$P(X = k) = (1 - p)^{k-1} \cdot p$$

- **Complement Rule:**

$$P(\text{At Least One}) = 1 - P(\text{None})$$

- **Common Mistake:** Forgetting to subtract from 1 when using the complement method.

Try Solving These!

1. Find the probability that the first boost occurs on the **seventh action**, given that the probability of a boost is $p = 0.2$.
2. Calculate the probability of **exactly three boosts** in the first **ten actions**, assuming $p = 0.3$.
3. What is the probability of **at least one boost** in the first **five actions**, if $p = 0.25$?
4. If the probability of a boost is $p = 0.4$, how many actions are expected before the first boost occurs? (Hint: Use the expected value formula for the geometric distribution.)
5. A player performs 12 actions, and the probability of a boost is $p = 0.2$. What is the probability of **at most two boosts**?

Note: Use the following formulas to solve:

- **Geometric Distribution:** $P(X = k) = (1 - p)^{k-1} \cdot p$
- **Binomial Distribution:** $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$
- **Complement Rule:** $P(\text{At Least One}) = 1 - P(\text{None})$
- **Expected Value (Geometric):** $E(X) = \frac{1}{p}$

Try Solving These!

1. A game developer wants to ensure that at least one boost occurs within the first 10 actions with a probability of 95%. What should the probability of a boost (p) be set to? (Hint: Use the complement rule and solve for p.)
2. Suppose the number of boosts occurring in a game follows a Poisson distribution with an average rate of $\lambda = 4$ per hour. What is the probability that in a 30-minute interval, there are either 2 or 3 boosts?
3. A player has a 20% chance of getting a special item after each level. If they play 8 levels, what is the probability that they get the special item in exactly 3 of those levels, given that they got the item at least once? (Hint: Use conditional probability.)
4. A bag contains 15 items, 5 of which are rare. If a player randomly selects 4 items without replacement, what is the probability that they get exactly 2 rare items? (Hint: Use the hypergeometric distribution.)
5. A call center receives an average of 6 calls per hour. What is the probability that in the next hour, they receive more than 8 calls? (Hint: Use the Poisson distribution and the complement rule.)

Problem 1 (b) (i)

Problem (b) (i)

Suppose the probability that an action will be boosted is p, where $0 < p < 1$.
(i) Explain why the probability that the first boost occurs on the xth action is

$$p(1-p)^{x-1}.$$

[1 Point]

Solutions

Solution

Step 1: Understanding the scenario

- For the first boost to occur on the xth action:
  - First $x - 1$ actions must not be boosted
  - The xth action must be boosted

Step 2: Computing the probability

- Probability of no boost on each action $= 1 - p$
- Probability of no boost for first $x - 1$ actions $= (1 - p)^{x-1}$
- Probability of boost on xth action $= p$

Step 3: Applying multiplication rule

$$P(X = x) = p(1 - p)^{x-1}$$

This is the probability mass function of a geometric distribution.

Alternative Approach

Step 1: Recognizing the Geometric Distribution

- This scenario follows a geometric distribution
- Models number of trials until first success
- Each trial is independent with probability p

Step 2: Using the Memoryless Property

- Past failures don't affect future probability
- PMF of geometric distribution: $P(X = x) = p(1 - p)^{x-1}$

Examiner's Expectation

Method (M1): [1 mark]

- Clear explanation of why $x-1$ failures must precede success
- Understanding of independence of trials
- Correct application of multiplication rule

Final Answer: $P(X=x)=p(1-p)^{x-1}$

Critical Points

Key Concepts:

- Geometric distribution models first success
- Independence of trials is crucial
- Multiplication rule for probability
- Memoryless property

Common Mistakes:

- Confusing with binomial distribution
- Incorrect order of multiplication
- Forgetting independence assumption

Bonus Exploration

Properties of Geometric Distribution:

- Expected value: $E(X)=\frac{1}{p}$
- Variance: $Var(X)=\frac{1-p}{p^2}$
- Memoryless property: $P(X>m+n|X>m)=P(X>n)$

Applications:

- Quality control: Number of items until first defect
- Gaming: Number of attempts until first win
- Research: Number of trials until first success

Quick Revision

Key Points:

- Geometric Distribution PMF: $P(X = x) = p(1-p)^{x-1}$
- Models first success in independent trials
- Each trial has constant probability p
- Trials are independent

Practice Problems

1. If $p = 0.3$, find the probability that the first boost occurs on:
   - The 4th action
   - The 7th action
2. For $p = 0.25$, calculate:
   - Expected number of actions until first boost
   - Probability of getting first boost within 5 actions

Challenge Problems

1. Prove that the sum of probabilities $\sum_{x=1}^{\infty} p(1-p)^{x-1} = 1$
2. Find the value of p if the probability of getting the first boost within 5 actions is 0.8
3. If $p = 0.2$, what is the probability that the first boost occurs after the 10th action?

Problem 1 (b)(ii)

Problem (b) (ii)

Let X be the number of actions until the first boost occurs.
(ii) Hence, write down an expression, using sigma notation, for $E(X)$ in terms of x and p.

$$E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1}$$

[1 Point]

Solutions

Solution

Step 1: Understanding Expected Value

- Expected value is the sum of each possible value multiplied by its probability
- For a discrete random variable: $E(X) = \sum x \cdot P(X = x)$

Step 2: Applying to Geometric Distribution

- $P(X = x) = p(1-p)^{x-1}$ is the probability of first success on xth trial
- Each value x is multiplied by its probability
- Sum extends to infinity as trials can continue indefinitely

Step 3: Writing the Expression

$$E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1}$$

Alternative Approach

Direct Formula Approach:

- For geometric distribution, $E(X) = \frac{1}{p}$
- This can be proven by evaluating the infinite sum:

$$\sum_{x=1}^{\infty} xp(1-p)^{x-1} = \frac{1}{p}$$

Examiner's Expectation

Method (M1): [1 mark]

- Correct use of sigma notation
- Proper expression for geometric probability
- Multiplication by x inside summation

Answer: $E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1}$

Critical Points

Key Concepts:

- Expected value formula for discrete random variables
- Geometric distribution probability mass function
- Infinite series representation

Common Mistakes:

- Forgetting to multiply by x inside the summation
- Using finite upper limit instead of infinity
- Incorrect power in $(1-p)$ term

Bonus Exploration

Expected Value Properties:

- For geometric distribution: $E(X) = \frac{1}{p}$
- This means average number of trials until success is reciprocal of success probability
- Example: If $p = 0.2$, expect first success after 5 trials on average

Series Evaluation:

- The infinite series can be evaluated using calculus techniques
- Related to power series and derivatives
- Demonstrates connection between probability and calculus

Quick Revision

Key Points:

- Expected value: $E(X) = \sum_{x=1}^{\infty} xp(1-p)^{x-1}$
- Simplified formula: $E(X) = \frac{1}{p}$
- Represents average number of trials until first success

Practice Problems

1. If $p = 0.25$, find:
   - Expected number of trials until first success
   - Probability of success within the expected number of trials
2. For $p = 0.4$, calculate:
   - $E(X)$
   - $P(X \leq E(X))$

Challenge Problems

1. Prove that $E(X) = \frac{1}{p}$ using the infinite series
2. If the expected number of trials is 8, what is the value of p?
3. Show that $P(X > E(X)) > 0.3$ for any value of p

Problem 1 (c)(i)

Problem (c)(i)

Consider the sum of an infinite geometric sequence, with first term a and common ratio r ($|r| < 1$):

$$a + ar + ar^2 + ar^3 + \dots$$

which sums to

$$\frac{a}{1-r}$$

(c)(i) By differentiating both sides of the above equation with respect to r, find an expression for

$$\sum_{n=1}^{\infty} nar^{n-1}$$

in terms of a and r. **[3 Points]**

Solutions

Detailed Solution

Step 1: Set up the infinite geometric series

- Let $S = a + ar + ar^2 + ar^3 + \dots$
- This can be written as $S = \sum_{n=1}^{\infty} ar^{n-1}$
- Given that $S = \frac{a}{1-r}$ for $|r| < 1$

Step 2: Differentiate both sides with respect to r

- LHS: $\frac{d}{dr} \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a(n-1)r^{n-2}$
- RHS: $\frac{d}{dr}(\frac{a}{1-r}) = \frac{a}{(1-r)^2}$

Step 3: Equate both sides

$$\sum_{n=1}^{\infty} a(n-1)r^{n-2} = \frac{a}{(1-r)^2}$$

Final Answer:

$$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$$

Alternative Approach

Using Power Series:

- Consider the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$
- Multiply both sides by a: $a \sum_{n=0}^{\infty} r^n = \frac{a}{1-r}$
- Differentiate: $a \sum_{n=1}^{\infty} nr^{n-1} = \frac{a}{(1-r)^2}$

Examiner's Expectation

Method (M1): [1 mark]

- Correct identification of series
- Attempt to differentiate both sides

Accuracy (A1): [1 mark]

- Correct differentiation of $\frac{a}{1-r}$
- Obtaining $\frac{a}{(1-r)^2}$

Final Answer (A1): [1 mark]

$$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$$

Critical Points

Key Concepts:

- Infinite geometric series convergence: $|r| < 1$
- Differentiation of power series
- Chain rule application

Common Mistakes:

- Forgetting the convergence condition $|r| < 1$
- Incorrect differentiation of $\frac{a}{1-r}$
- Mixing up powers in the series

Bonus Exploration

Applications in Calculus:

- Power series representations
- Taylor series connections
- Generating functions in probability

Further Exploration:

- What happens when we differentiate twice?
- How does this relate to variance in probability?
- Applications in physics and engineering

Practice Problems

1. Find $\sum_{n=1}^{\infty} n^2 r^{n-1}$ by differentiating twice.
2. If $|r| = \frac{1}{2}$, evaluate $\sum_{n=1}^{\infty} nr^{n-1}$.
3. Show that $\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$.

Challenge Problems

1. Prove that $\sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3}$.
2. Find a general formula for $\sum_{n=1}^{\infty} n^k r^{n-1}$ for any positive integer k.
3. How does this relate to the moment-generating function in probability theory?

Problem 1 (c)(ii)

Problem (c)(ii)

Hence, show that

$$E(X) = \frac{1}{p}$$

[2 Points]

Solutions

Solution Using Previous Result

Step 1: Recall the result from part (c)(i)

$$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$$

Step 2: Connect to geometric distribution

- For geometric distribution: $P(X = n) = p(1-p)^{n-1}$
- Compare with our series where:
  - $a = p$
  - $r = 1 - p$

Step 3: Apply the formula

$$E(X) = \sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Final Answer: Therefore, $E(X) = \frac{1}{p}$

Alternative Approach

Step 1: Write expectation definition

$$E(X) = \sum_{n=1}^{\infty} nP(X = n) = \sum_{n=1}^{\infty} np(1-p)^{n-1}$$

Step 2: Use power series result

$$\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}$$

Step 3: Substitute $r = 1 - p$

$$E(X) = p \cdot \frac{1}{(1-(1-p))^2} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

Examiner's Expectation

Method (M1): [1 mark]

- Correct identification: $a = p$ and $r = 1 - p$
- Proper substitution into the formula

Accuracy (A1): [1 mark]

- Correct algebraic manipulation
- Final answer: $E(X) = \frac{1}{p}$

Critical Points

Key Concepts:

- Connection between series and expectation
- Proper substitution of parameters
- Algebraic manipulation skills

Common Mistakes:

- Incorrect parameter substitution
- Errors in algebraic simplification
- Not showing clear steps

Bonus Exploration

Interpretation of $E(X) = \frac{1}{p}$:

- Average number of trials until success
- Inverse relationship with probability
- Real-world applications

Examples:

- If $p = 0.2$, expect 5 trials on average
- If $p = 0.5$, expect 2 trials on average
- As p decreases, expected trials increase

Practice Problems

1. If $E(X) = 8$, find the value of p
2. Calculate $E(X)$ when $p = 0.25$
3. Find p if the average number of trials is 10

Challenge Problems

1. Prove that $Var(X) = \frac{1-p}{p^2}$
2. Find $E(X^2)$ using the series approach
3. Show that $P(X > E(X)) > 0.3$ for all p

Problem 1 (d)

Problem (d)

It can be shown that

$$\text{Var}(X) = \frac{1-p}{p^2}$$

Find $E(X)$ and $\text{Var}(X)$ when $p = 0.2$. **[2 Points]**

Solutions

Detailed Solution

Step 1: Calculate $E(X)$

- For geometric distribution: $E(X) = \frac{1}{p}$
- Substitute $p = 0.2$:
$$E(X) = \frac{1}{0.2} = 5$$

Step 2: Calculate Var(X)

- Given formula: Var$(X) = \frac{1-p}{p^2}$
- Substitute $p = 0.2$:
$$\text{Var}(X) = \frac{1-0.2}{(0.2)^2} = \frac{0.8}{0.04} = 20$$

Final Answer:

- $E(X) = 5$
- Var$(X) = 20$

Interpretation

Meaning of Results:

- On average, it takes 5 trials to get the first success
- The variance of 20 indicates considerable spread around this mean
- Standard deviation $= \sqrt{20} \approx 4.47$ trials

Examiner's Expectation

Accuracy (A1): [1 mark]

- Correct calculation of $E(X) = 5$

Accuracy (A1): [1 mark]

- Correct calculation of Var$(X) = 20$

Total: [2 marks]

Critical Points

Key Concepts:

- Direct substitution into formulas
- Careful arithmetic calculations
- Understanding decimal arithmetic

Common Mistakes:

- Calculation errors with decimals
- Forgetting to simplify fractions
- Mixing up formulas for $E(X)$ and $\text{Var}(X)$

Bonus Exploration

Relationship Between $E(X)$ and Var(X):

- As p decreases:
  - $E(X)$ increases
  - $\text{Var}(X)$ increases more rapidly
- $\text{Var}(X) > E(X)$ for all $p < 0.5$
- Coefficient of variation $= \frac{\sqrt{\text{Var}(X)}}{E(X)} = \sqrt{\frac{1-p}{p}}$

Practice Problems

1. Calculate $E(X)$ and $\text{Var}(X)$ when:
   - $p = 0.25$
   - $p = 0.5$
2. Find p if $E(X) = 10$
3. Calculate the standard deviation when $p = 0.2$

Challenge Problems

1. Prove that $\text{Var}(X) > E(X)$ when $p < 0.5$
2. Find the value of p that minimizes the coefficient of variation
3. If $E(X) = 4$, find the probability that X exceeds twice its mean

Problem 1 (e)

Problem (e)

In the designer's second model, the initial probability that an action is boosted is 0.25, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.25. After an action has been boosted, the probability resets to 0.25 for the next action.
(e) Show that the probability that the first boost occurs on the third action is 0.28125. **[2 Points]**

Solutions

Solution with Tree Diagram

Step 1: Analyze the changing probabilities

- Initial probability (1st action): $p_1 = 0.25$
- After one failure (2nd action): $p_2 = 0.50$
- After two failures (3rd action): $p_3 = 0.75$

Step 2: Calculate required probabilities

- P(No boost on 1st) $= 1 - 0.25 = 0.75$
- P(No boost on 2nd — No boost on 1st) $= 1 - 0.50 = 0.50$
- P(Boost on 3rd — No boost on 1st and 2nd) $= 0.75$

Step 3: Apply multiplication rule

$$P(\text{First boost on 3rd}) = 0.75 \times 0.50 \times 0.75 = 0.28125$$

Alternative Approach

Using Conditional Probability:

$$\begin{aligned}P(X=3) &= P(\text{No boost on 1st}) \times P(\text{No boost on 2nd}|\text{No boost on 1st}) \\ &\quad \times P(\text{Boost on 3rd}|\text{No boost on 1st and 2nd}) \\ &= (1-0.25)\times(1-0.50)\times 0.75 \\ &= 0.75\times 0.50\times 0.75 \\ &= 0.28125\end{aligned}$$

Examiner's Expectation

Method (M1): [1 mark]

- Recognition of changing probabilities
- Correct identification of 0.75, 0.50, and 0.75

Accuracy (A1): [1 mark]

- Correct multiplication
- Final answer: 0.28125

Critical Points

Key Concepts:

- Changing probability after each failure
- Conditional probability
- Multiplication rule for independent events

Common Mistakes:

- Not accounting for probability increase
- Incorrect multiplication order
- Calculation errors with decimals

Bonus Exploration

Comparison with Regular Geometric Distribution:

- Regular geometric: constant probability
- This model: increasing probability
- Effect on expected value
- Impact on variance

Applications:

- Gaming mechanics
- Adaptive systems
- Progressive difficulty

Practice Problems

1. Calculate the probability of first boost on:
   - Fourth action
   - Second action
2. Find the probability of no boost in first three actions
3. Compare with regular geometric distribution $(p = 0.25)$

Challenge Problems

1. Find the expected number of actions until first boost
2. Compare probabilities if increase is 0.20 instead of 0.25
3. Determine probability of at least one boost in first three actions

Problem 1 (f)

Problem (f)

Let Y be the number of actions until the first boost occurs.
Explain why $Y \leq 4$. **[1 Point]**

Solutions

Solution

Step 1: Understanding the probability dynamics

- The initial probability of an action being boosted is 0.25.
- Each time an action is not boosted, the probability of the next action being boosted increases by 0.25.
- After an action is boosted, the probability resets to 0.25.

Step 2: Determine the upper bound on Y

- If the first action is not boosted, the probability of a boost increases to 0.50 for the second action.
- If the second action is also not boosted, the probability of a boost increases to 0.75 for the third action.
- If the third action is still not boosted, the probability of a boost reaches 1 on the fourth action.
- This means that the first boost must occur by the fourth action.

Final Answer:

$$Y \leq 4.$$

Alternative Solution

Step 1: Identify the probability condition

- The initial probability of an action being boosted is 0.25.
- Each time an action is not boosted, the probability of the next action being boosted increases by 0.25.
- The probability cannot exceed 1, as it represents a certainty.

Step 2: Determine the upper bound on Y

- If an action is not boosted for three consecutive attempts, the probability for the fourth attempt reaches:

$$0.25 + 0.25 + 0.25 + 0.25 = 1.$$

- This means that a boost must occur by the fourth action at the latest.
- Therefore, Y cannot exceed 4.

Final Answer:

$$Y \leq 4.$$

Examiner's Expectation

Condition: After three actions that are not boosted, the probability that the next action is boosted becomes 1, meaning $p = 1$ (certainty).
Reasoning (R1):

- Acceptable: "When $Y = 4$, the probability of a boost is 1."
- Not acceptable: "On the 4th action, the probability of a boost is 1" (unless it explicitly mentions that the previous actions were not boosted).

Final Conclusion:

$$Y \leq 4.$$

[1 Point]

Critical Points

Key Concepts:

- Probability increases by 0.25 after each failure.
- The probability reaches 1 (certainty) on the fourth action if no boost occurs earlier.
- The first boost must occur by the fourth action.

Common Mistakes:

- Forgetting that the probability resets after a boost.
- Misinterpreting the probability increase as constant across all actions.
- Ignoring the upper limit of probability ($p \leq 1$).

Bonus Exploration

Comparison with Regular Geometric Distribution:

- Regular geometric: constant probability of success.
- This model: increasing probability after each failure.
- Effect on expected value and variance.

Applications:

- Adaptive systems in gaming.
- Progressive difficulty models.
- Real-world scenarios with increasing probabilities.

Practice Problems

1. Explain why $Y \leq 5$ if the probability increase is 0.20 instead of 0.25.
2. Calculate the probability of no boost in the first three actions.
3. Compare the expected value of Y for this model with a regular geometric distribution ($p = 0.25$).

Challenge Problems

1. Prove that $Y \leq 4$ for any probability increase $\Delta p \leq 0.25$.
2. Find the expected value of Y for this model.
3. Determine the variance of Y for this model.

Problem 1 (g) (i)

Problem (g) (i)

The following table shows the probability distribution of Y:

y	1	2	3	4
$P(Y = y)$	0.25	m	0.28125	n

(i) Find the value of m and the value of n. **[2 Points]**

Solutions

Solution

Step 1: Calculate m (Probability of boost on the 2nd action)
The first action must not be boosted, and the second action must be boosted:

$$P(\text{Boost on 2nd action}) = P(\text{No boost on 1st}) \times P(\text{Boost on 2nd}).$$

$$P(Y = 2) = m = 0.75 \times 0.5 = 0.375.$$

Step 2: Calculate n (Probability of boost on the 4th action)
The first three actions must not be boosted, and the fourth action must be boosted:

$$P(\text{Boost on 4th action}) = P(\text{No boost on 1st}) \times P(\text{No boost on 2nd}) \times P(\text{No boost on 3rd}) \times P(\text{Boost on 4th}).$$

$$P(Y = 4) = n = 0.75 \times 0.5 \times 0.25 \times 1 = 0.09375.$$

Step 3: Verify the total probability
The sum of all probabilities must equal 1:

$$P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) = 1.$$

$$0.25 + 0.375 + 0.28125 + 0.09375 = 1.$$

Final Answer:

$$m = 0.375, \quad n = 0.09375.$$

Alternative Solution

Step 1: Calculate m using the multiplication rule:

$$P(Y = 2) = P(\text{No boost on 1st}) \times P(\text{Boost on 2nd}).$$

$$m = 0.75 \times 0.5 = 0.375.$$

Step 2: Calculate n using the total probability rule:

$$n = 1 - P(Y = 1) - P(Y = 2) - P(Y = 3).$$

$$n = 1 - 0.25 - 0.375 - 0.28125 = 0.09375.$$

Final Answer:

$$m = 0.375, \quad n = 0.09375.$$

Examiner's Expectation

Points Allocation:

- Correct calculation of $m = 0.375$ earns **A1**.
- Correct calculation of $n = 0.09375$ earns **A1**.

Total: 2 Points.

Critical Points

Key Concepts:

- Use the multiplication rule for independent events.
- Ensure the total probability sums to 1.
- Verify calculations for m and n using the total probability rule.

Common Mistakes:

- Forgetting to multiply probabilities correctly.
- Not verifying that the total probability equals 1.
- Misinterpreting the probability increase dynamics.

Bonus Exploration

Applications of Probability Distributions:

- Modeling adaptive systems with changing probabilities.
- Real-world scenarios where probabilities increase or decrease dynamically.
- Comparing this model with a regular geometric distribution.

Practice Problems

1. Verify the total probability for $P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) = 1$.
2. Calculate $P(Y = 5)$ if the probability increase is 0.20 instead of 0.25.
3. Compare the expected value of Y for this model with a regular geometric distribution ($p = 0.25$).

Challenge Problems

1. Prove that the total probability for any n-step model equals 1.
2. Find the expected value of Y for this model.
3. Determine the variance of Y for this model.

Problem 1 (g) (ii)

Problem (g) (ii)

The following table shows the probability distribution of Y:

y	1	2	3	4
$P(Y = y)$	0.25	m	0.28125	n

(ii) Show that $E(Y) = 2.21875$. **[2 Points]**

Solutions

Solution

The expected value $E(Y)$ is calculated using the formula:

$$E(Y) = \sum y \cdot P(Y = y)$$

Step 1: Substituting the given values:

$$E(Y) = (1 \times 0.25) + (2 \times 0.375) + (3 \times 0.28125) + (4 \times 0.09375)$$

Step 2: Perform the calculations step by step:

$$E(Y) = 0.25 + 0.75 + 0.84375 + 0.375$$

Step 3: Add the results:

$$E(Y) = 2.21875$$

Final Answer:

$$E(Y) = 2.21875$$

Examiner's Expectation

Points Allocation:

- Correct substitution into the formula:

  $$E(Y) = (1 \times 0.25) + (2 \times 0.375) + (3 \times 0.28125) + (4 \times 0.09375)$$

  earns **M1**.

- Correct stepwise computation and final answer:

  $$E(Y) = 2.21875$$

  earns **A1**.

Total: 2 Points.

Critical Points

Key Concepts:

- Expected value is the weighted average of outcomes.
- Ensure all probabilities sum to 1 before calculating $E(Y)$.
- Perform stepwise calculations to avoid errors.

Common Mistakes:

- Forgetting to multiply outcomes by their probabilities.
- Skipping steps in the calculation, leading to errors.
- Using incorrect values for m or n.

Bonus Exploration

Applications of Expected Value:

- Decision-making in uncertain scenarios.
- Modeling adaptive systems with changing probabilities.
- Comparing this model with a regular geometric distribution.

Practice Problems

1. Verify the calculation of $E(Y)$ using the given table.
2. Calculate $E(Y)$ if $P(Y = 4)$ is changed to 0.1.
3. Compare the expected value of Y for this model with a regular geometric distribution ($p = 0.25$).

Challenge Problems

1. Prove that the expected value $E(Y)$ for this model is always less than 3.
2. Find the variance Var(Y) for this model.
3. Determine the expected value if the probability increase is 0.20 instead of 0.25.

Problem 1 (g) (iii)

Problem (g) (iii)

The following table shows the probability distribution of Y:

y	1	2	3	4
$P(Y = y)$	0.25	m	0.28125	n

(iii) Find Var(Y). **[2 Points]**

Solutions

Detailed Solution

Step 1: Calculate $E(Y^2)$

$$\begin{aligned} E(Y^2) &= \sum y^2 \cdot P(Y = y) \\ &= (1^2 \times 0.25) + (2^2 \times 0.375) + (3^2 \times 0.28125) + (4^2 \times 0.09375) \\ &= (1 \times 0.25) + (4 \times 0.375) + (9 \times 0.28125) + (16 \times 0.09375) \\ &= 0.25 + 1.5 + 2.53125 + 1.5 \\ &= 5.78125 \end{aligned}$$

Step 2: Recall $E(Y) = 2.21875$ from part (ii)
Step 3: Calculate Var(Y) using the formula

$$\begin{aligned} \text{Var}(Y) &= E(Y^2) - [E(Y)]^2 \\ &= 5.78125 - (2.21875)^2 \\ &= 5.78125 - 4.924316 \\ &= 0.856934 \end{aligned}$$

Final Answer:

$$\text{Var}(Y) = 0.857 \text{ (3 d.p.)}$$

Alternative Solution Using Calculator

Using a calculator:

- Input the values and probabilities
- Calculate variance directly
- Verify result matches the manual calculation

Final Answer:

$$\text{Var}(Y) = 0.857$$

Examiner's Expectation

Points Allocation:

- **A1:** Correct calculation of $E(Y^2)$
- **A1:** Correct final variance calculation

Acceptable Answers:

- Exact value: 0.856934
- Rounded to 3 d.p.: 0.857

Total: 2 Points

Critical Points

Key Concepts:

- Variance formula: $\text{Var}(Y) = E(Y^2) - [E(Y)]^2$
- Square each outcome before multiplying by probability
- Use exact values throughout calculation

Common Mistakes:

- Forgetting to square outcomes when calculating $E(Y^2)$
- Using rounded values in intermediate steps
- Incorrect application of variance formula

Bonus Exploration

Interpretation of Variance:

- Measures spread of outcomes around mean
- Standard deviation $= \sqrt{\text{Var}(Y)} \approx 0.926$
- Compare with regular geometric distribution

Applications:

- Risk assessment in gaming
- Predictability of outcomes
- System reliability analysis

Practice Problems

1. Calculate the standard deviation of Y
2. Find Var$(2Y + 1)$
3. Compare this variance with a regular geometric distribution

Challenge Problems

1. Prove that Var(Y) is less than 1
2. Find $P(|Y - E(Y)| > 2\sigma)$
3. Calculate the coefficient of variation

Problem 1 (h) (i)

Problem (h) (i)

Use the expression given in (c)(ii) to find the value of p for which $E(X) = E(Y)$. **[1 Point]**

Solution

Solution

The expected value of Y is given as:

$$E(Y) = 2.21875$$

From (c)(ii), the expected value of X is:

$$E(X) = \frac{1}{p}$$

To find the value of p for which $E(X) = E(Y)$, we equate the two expressions:

$$\frac{1}{p} = 2.21875$$

Step 1: Solve for p:

$$p = \frac{1}{2.21875}$$

Step 2: Perform the calculation:

$$p \approx 0.45$$

Final Answer:

$$p \approx 0.45$$

Examiner's Expectation

Points Allocation:

- Correctly equating $E(X) = E(Y)$ and substituting the expressions earns **M1**.
- Correct calculation of $p \approx 0.45$ earns **A1**.

Total: 1 Point.

Critical Points

Key Concepts:

- The expected value of a geometric random variable is $E(X) = \frac{1}{p}$.
- Equating $E(X)$ and $E(Y)$ allows us to solve for p.
- Ensure accurate calculations when solving for p.

Common Mistakes:

- Forgetting to invert $E(Y)$ when solving for p.
- Rounding errors in intermediate steps.
- Misinterpreting the relationship between $E(X)$ and $E(Y)$.

Bonus Exploration

Applications of Expected Value:

- Decision-making in uncertain scenarios.
- Comparing different probability models.
- Understanding the relationship between geometric and adaptive distributions.

Practice Problems

1. Verify the calculation of p for $E(X) = 3$.
2. Find p if $E(Y) = 3.5$.
3. Compare the expected values of X and Y for $p = 0.4$.

Challenge Problems

1. Prove that p must always satisfy $0 < p \leq 1$ for $E(X) = E(Y)$.
2. Find the variance of X when $p = 0.45$.
3. Determine the value of p for which $E(X) = 2E(Y)$.

Problem 1 (h) (ii)

Problem (h) (ii)

Find Var(X) for this value of p. **[1 Point]**

Solutions

Solution

Step 1: Recall the formula for variance of a geometric distribution

$$\text{Var}(X) = \frac{1-p}{p^2}$$

Step 2: Substitute $p = 0.45$ from part (h)(i)

$$\text{Var}(X) = \frac{1-0.45}{(0.45)^2}$$

Step 3: Calculate step by step

$$\begin{aligned}\text{Var}(X) &= \frac{0.55}{0.2025} \\ &= 2.716\end{aligned}$$

Final Answer:

$$\text{Var}(X) = 2.716$$

Alternative Solution Using Calculator

Using a calculator:

- Input the formula $\frac{1-p}{p^2}$
- Substitute $p = 0.45$
- Calculate directly

Final Answer:

$$\text{Var}(X) = 2.716$$

Examiner's Expectation

Points Allocation:

- **A1:** Correct answer of 2.716 (or 2.72)
- Condone 2.7

Note: Answer must use the value of p found in part (h)(i).
Total: 1 Point

Critical Points

Key Concepts:

- Variance formula for geometric distribution: $\text{Var}(X) = \frac{1-p}{p^2}$
- Use exact value of p from previous part
- Maintain precision in calculations

Common Mistakes:

- Using incorrect variance formula
- Rounding p too early in calculations
- Forgetting to square p in denominator

Bonus Exploration

Comparing Variances:

- $\text{Var}(X) = 2.716$ for geometric distribution
- $\text{Var}(Y) = 0.857$ for adaptive model
- The geometric distribution shows greater spread

Applications:

- Risk assessment in gaming mechanics
- Reliability analysis
- Quality control processes

Practice Problems

1. Calculate Var(X) for $p = 0.5$
2. Find the standard deviation of X
3. Compare Var(X) with Var(Y)

Challenge Problems

1. Prove that Var$(X) >$ Var(Y) for this value of p
2. Find Var$(2X + 1)$
3. Determine the coefficient of variation for X

Problem 1 (h) (iii)

Problem (h) (iii)

Hence, determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. **[1 Point]**

Solutions

Solution

Step 1: Compare the variances

- Model 1 (Geometric): Var$(X) = 2.716$
- Model 2 (Adaptive): Var$(Y) = 0.857$

Step 2: Interpret the variances

- A smaller variance indicates less spread in the outcomes
- Less spread means more predictable, consistent experiences
- Var$(Y) <$ Var(X)

Final Answer: Model 2 (the adaptive model) provides a more consistent experience for the player because it has a smaller variance (0.857 compared to 2.716), meaning the number of actions until a boost occurs is more predictable.

Alternative Explanation

Statistical Reasoning:

- The standard deviation of Model 1: $\sqrt{2.716} \approx 1.648$
- The standard deviation of Model 2: $\sqrt{0.857} \approx 0.926$
- The smaller standard deviation in Model 2 indicates less deviation from the expected value
- Therefore, Model 2 provides more consistent timing between boosts

Examiner's Expectation

Points Allocation:

- **R1:** Correct conclusion with valid reasoning based on comparison of variances

Required Elements:

- Explicit comparison of variances
- Clear link between smaller variance and consistency
- Correct identification of Model 2 as more consistent

Total: 1 Point

Critical Points

Key Concepts:

- Variance measures spread/variability of outcomes
- Lower variance indicates more consistent experiences
- Relationship between variance and predictability

Common Mistakes:

- Incorrect interpretation of variance
- Not providing clear reasoning
- Confusing consistency with effectiveness

Bonus Exploration

Game Design Implications:

- Balance between predictability and randomness
- Player engagement and satisfaction
- Adaptive systems in modern games

Statistical Applications:

- Coefficient of variation comparison
- Risk assessment in gaming mechanics
- Player experience optimization

Practice Problems

1. Compare the coefficients of variation for both models
2. Analyze the impact of changing p on consistency
3. Explore other measures of variability

Challenge Problems

1. Design a third model that provides even more consistency
2. Calculate the probability of extreme deviations in both models
3. Analyze the trade-off between consistency and unpredictability

Problem 2 : [28 Points]

Problem Description

- This problem investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.
- The curve $y = x^4 - 4x^3 + 7x$ has points of inflexion at B and C.
- The line passing through B and C intersects the curve again at points A and D, as shown in the given graph.

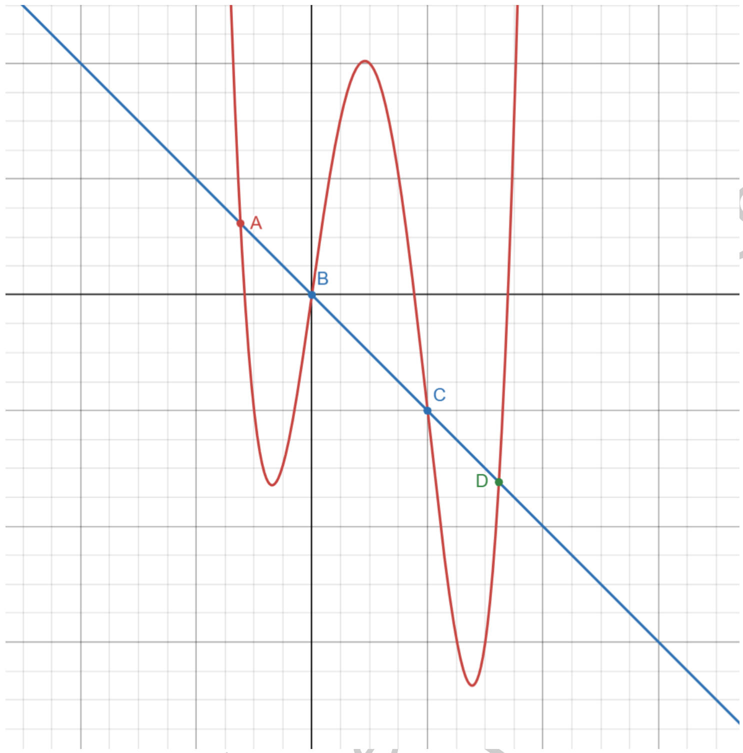

Figure 1: Graph showing the curve

Now, Solve the following Problems based on above information!

Problem (a)

Find $\frac{d^2y}{dx^2}$. **[3 Points]**

Problem (b)

Find the coordinates of B and C. **[4 Points]**

Problem (c)

Show that the equation of the line through points B and C is $y = -x$. **[2 Points]**

Problem (d)

Find, correct to three decimal places, the x-coordinate of point D. **[2 Points]**

Problem (e)

For the general curve $y = x^4 - mx^3 + nx$, where $m, n \in \mathbb{R}$ and $m > 0$. Find the x-coordinates of the two points of inflection in terms of m. **[3 Points]**

Problem (f)

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x-coordinate of point A, and similarly for x_B, x_C, and x_D. It is given that $x_A < x_B < x_C < x_D$.

Problem (f)(i)

Write down the coordinates of B. **[1 Point]**

Problem (f)(ii)

Find, in terms of m and n, the coordinates of C. **[2 Points]**

Problem (g)

Show that the equation of the line through points B and C is

$$y = (-\frac{m^3}{8} + n)x.$$

[2 Points]

Problem (h)

Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. **[7 Points]**

Problem (i)

Hence, find the exact value of

$$\frac{x_B - x_A}{x_C - x_B}.$$

[2 Points]

Solutions

Solution 2 (a)

Solution 2 (a)

We need to find the second derivative of the given function:

$$y = x^4 - 4x^3 + 7x$$

Step 1: Compute the first derivative

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 4x^3 + 7x)$$

$$= 4x^3 - 12x^2 + 7$$

Step 2: Compute the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - 12x^2 + 7)$$

$$= 12x^2 - 24x$$

Final Answer:

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

Solution 2 (a)

Solution 2 (a)

We need to find the second derivative of the given function:

$$y = x^4 - 4x^3 + 7x$$

Step 1: Compute the first derivative

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 4x^3 + 7x)$$

$$= 4x^3 - 12x^2 + 7$$

Step 2: Compute the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - 12x^2 + 7)$$

$$= 12x^2 - 24x$$

Final Answer:

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

Solution 2 (b)

Solution 2 (b)

We need to find the coordinates of the points of inflexion, B and C.
Step 1: Set the second derivative to zero

$$12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$$x = 0 \text{ or } x = 2$$

Step 2: Find the corresponding y-coordinates Substituting $x = 0$ into the original equation:

$$y = (0)^4 - 4(0)^3 + 7(0) = 0$$

So, one inflection point is $B(0, 0)$.
Substituting $x = 2$:

$$y = (2)^4 - 4(2)^3 + 7(2)$$
$$= 16 - 32 + 14 = -2$$

So, the second inflection point is $C(2, -2)$.
Final Answer:

$$B(0, 0), \quad C(2, -2)$$

Solution 2 (c)

Solution 2 (c)

We need to find the equation of the line passing through points B and C.
Step 1: Find the slope of BC Using the formula for the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting $B(0, 0)$ and $C(2, -2)$:

$$m = \frac{-2 - 0}{2 - 0} = \frac{-2}{2} = -1$$

Step 2: Use point-slope form The equation of a line is:

$$y - y_1 = m(x - x_1)$$

Substituting $B(0, 0)$ and $m = -1$:

$$y - 0 = -1(x - 0)$$

$$y = -x$$

Final Answer:

$$y = -x$$

Solution 2 (d)

Solution 2 (d)

We need to find the x-coordinate of point D to three decimal places.
Step 1: Solve for intersection points of the line and the curve We solve $y = -x$ in the quartic equation:

$$x^4 - 4x^3 + 7x = -x$$

$$x^4 - 4x^3 + 8x = 0$$

$$x(x^3 - 4x^2 + 8) = 0$$

Since we already have roots at $x = 0$ and $x = 2$, we solve:

$$x^3 - 4x^2 + 8 = 0$$

Using numerical methods or graphing, the largest positive root is found to be:

$$x_D \approx 3.512$$

Final Answer:

$$x_D = 3.512$$

Solution 2 (e)

Solution 2 (e)

We consider the general quartic function:

$$y = x^4 - mx^3 + nx$$

Step 1: Computing the second derivative

$$\frac{dy}{dx} = 4x^3 - 3mx^2 + n$$

$$\frac{d^2y}{dx^2} = 12x^2 - 6mx$$

Setting $\frac{d^2y}{dx^2} = 0$ to find points of inflexion:

$$12x^2 - 6mx = 0$$

$$6x(2x - m) = 0$$

$$x = 0, x = m/2$$

Final Answer: The x-coordinates of the two points of inflexion in terms of m are 0 and m/2.

Solution 2 (f)

Solution 2 (f)

(i) Finding the coordinates of B
From part (e), we found that the points of inflexion occur at:

$$x = 0 \quad \text{and} \quad x = \frac{m}{2}$$

Substituting $x = 0$ into the function:

$$y = (0)^4 - m(0)^3 + n(0) = 0$$

Thus, the coordinates of B are:

$$B(0, 0)$$

(ii) Finding the coordinates of C
Substituting $x = \frac{m}{2}$ into the function:

$$y = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right)$$

Simplifying each term:

$$y = \frac{m^4}{16} - m \cdot \frac{m^3}{8} + n \cdot \frac{m}{2}$$

$$y = \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}$$

$$y = \frac{m^4}{16} - \frac{2m^4}{16} + \frac{8nm}{16}$$

$$y = \frac{-m^4 + 8nm}{16}$$

Thus, the coordinates of C are:

$$C\left(\frac{m}{2}, \frac{8nm - m^4}{16}\right)$$

Final Answer: The coordinates of B and C are

$$B(0, 0), \quad C\left(\frac{m}{2}, \frac{8nm - m^4}{16}\right)$$

Solution 2 (g)

Solution 2 (g)

We need to find the equation of the line passing through points B and C.
Step 1: Finding the slope of line BC
Using the formula for the slope:

$$m_{\mathrm{BC}} = \frac{y_C - y_B}{x_C - x_B}$$

Substituting the coordinates:

$$m_{\mathrm{BC}} = \frac{\frac{8nm - m^4}{16} - 0}{\frac{m}{2} - 0}$$

$$m_{\mathrm{BC}} = \frac{\frac{8nm - m^4}{16}}{\frac{m}{2}}$$

$$m_{\mathrm{BC}} = \frac{(8nm - m^4)}{16} \times \frac{2}{m}$$

$$m_{\mathrm{BC}} = \frac{2(8nm - m^4)}{16m}$$

$$m_{\mathrm{BC}} = \frac{16nm - 2m^4}{16m}$$

$$m_{\mathrm{BC}} = \frac{16nm}{16m} - \frac{2m^4}{16m}$$

$$m_{\mathrm{BC}} = n - \frac{m^3}{8}$$

Step 2: Finding the equation of line BC
Using the point-slope form:

$$y - y_1 = m(x - x_1)$$

Substituting $B(0,0)$ and $m_{\mathrm{BC}} = n - \frac{m^3}{8}$:

$$y - 0 = \left(n - \frac{m^3}{8}\right)(x - 0)$$

$$y = \left(n - \frac{m^3}{8}\right)x$$

Final Answer:

$$y = \left(n - \frac{m^3}{8}\right)x$$

Solution 2 (h)

Solution 2 (h)

Step 1: Finding the equation for the x-coordinates
From previous results, the x-coordinates of the intersection points satisfy:

$$x^4 - mx^3 + nx = \left(n - \frac{m^3}{8}\right)x$$

Rearranging,

$$x^4 - mx^3 + nx + \frac{m^3}{8}x - nx = 0$$

$$x^4 - mx^3 - \frac{m^3}{8}x = 0$$

Since we already know $x = 0$ and $x = \frac{m}{2}$ are roots, the given quartic equation must be factorized as:
Thus,

$$x(x - \frac{m}{2})(x^2 - \frac{m}{2}x - \frac{m^2}{4}) = 0$$

So the quadratic equation becomes containing Point A & D:

$$x^2 - \frac{m}{2}x - \frac{m^2}{4} = 0$$

Step 2: Solving for x_A
Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-\frac{m}{2}) \pm \sqrt{(-\frac{m}{2})^2 + 4(\frac{m^2}{4})}}{2}$$

$$x = \frac{m}{4} \pm \frac{m\sqrt{5}}{4}$$

Since $x_A < x_B < x_C < x_D$, we choose:

$$x_A = \frac{m}{4} - \frac{m\sqrt{5}}{4}$$

Final Answer:

$$x_A = \frac{m}{4} - \frac{m\sqrt{5}}{4}$$

Solution 2 (i)

Solution 2 (i)

We need to compute the exact value of:

$$\frac{x_B - x_A}{x_C - x_B}$$

Step 1: Recall the values of x_A, x_B, x_C
From the previous part, we have:

$$x_A = \frac{m}{4} - \frac{m\sqrt{5}}{4}$$

$$x_B = 0$$

$$x_C = \frac{m}{2}$$

Step 2: Compute $x_B - x_A$

$$x_B - x_A = 0 - \left(\frac{m}{4} - \frac{m\sqrt{5}}{4}\right)$$

$$= \frac{m\sqrt{5} - m}{4}$$

Step 3: Compute $x_C - x_B$

$$x_C - x_B = \frac{m}{2} - 0 = \frac{m}{2}$$

Step 4: Compute the required fraction

$$\frac{x_B - x_A}{x_C - x_B} = \frac{\frac{m\sqrt{5}-m}{4}}{\frac{m}{2}}$$

$$= \frac{\sqrt{5} - 1}{2}$$

Final Answer:

$$\frac{x_B - x_A}{x_C - x_B} = \frac{\sqrt{5} - 1}{2}$$

Examiner's Expectation

Examiner's Expectation for Question 2

(a)

Points Allocation: - Award **M1** for correctly differentiating the function and obtaining at least two correct terms. - Award **A1** for the fully correct derivative. - Award **A1** for the correct second derivative.
Note: - Partial credit is given for attempting differentiation but missing one or more terms.

[3 Points]

(b)

Points Allocation: - Award **M1** for a valid attempt to find the x-coordinates by solving the second derivative equation. - Award **A1** for correctly finding both x-values. - Award **A1** for correctly identifying point B. - Award **A1** for correctly identifying point C.
Note: - Full credit is awarded only if both points are correctly identified. - Award **M0A0A1A0** for an unsupported answer of (0,0) as point B.

[4 Points]

(c)

Points Allocation: - Award **R1** for recognizing that the y-intercept is 0 by substituting into the equation. - Award **A1** for correctly calculating the gradient.
Note: - Award **A0FT** if an incorrect answer from (b) is used but correctly substituted. - Award at most **A1R0** for verifying without proper derivation.

[2 Points]

(d)

Points Allocation: - Award **M1** for correctly setting up and solving the equation. - Award **A1** for the correct x-value rounded to three decimal places.
Note: - Award **M1A0** for an unsupported answer of 2.43.

[2 Points]

(e)

Points Allocation: - Award **A1** for the first derivative with correct coefficients. - Award **A1** for the correct second derivative. - Award **A1** for solving for critical points.
Note: - Accept $0m$ instead of 0.

[3 Points]

Examiner's Expectation Continued...

(f)

Points Allocation: - Award **A1** for correctly identifying point B. - Award **A1** for the correct equation and simplification. - Award **A1** for expressing point C correctly.
Note: - The second **A1** is awarded for proper simplification in (g) as well.

[3 Points]

(g)

Points Allocation: - Award **M1** for an attempt to divide or apply the slope formula. - Award **A1** for obtaining the correct equation.
Note: - Award **A0FT** if an incorrect value is used but properly manipulated. - Do not award more than **M1A0** for verifying without proper derivation.

[2 Points]

(h)

Points Allocation: - Award **M1** for correctly setting up the equation in terms of roots. - Award **M1** for recognizing the factorization structure. - Award **A1** for the correct quadratic equation. - Award **M1** for using the quadratic formula. - Award **A1** for the correct root expression.
Note: - Condone $\pm$ in place of the minus sign if later correctly restated. - Award **M1M0A0M0A0M0A0** for attempting to verify rather than derive the result.

[7 Points]

Practice Problems

Practice Problem - 1

Problem 1

Maximum Points: 31
This question explores the mechanics of aiming at a target with random accuracy. Assume that when you aim at a circular board with radius 2, your throw is completely random, meaning that every point within the target is equally likely to be hit.

Let X represent the distance of the landing point from the center of the target.

(a)

(i) Determine the median value of X.

(ii) Show that the probability density function of X is given by $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$.

(iii) Compute $E(X)$.

[7 Points]

Define the score as $S = \frac{1}{X}$.

(b)

(i) Find the median of S.

(ii) Show that the probability density function of S is given by $g(s) = \frac{2}{s^3}$ for $s > \frac{1}{2}$.

(iii) Compute $E(S)$.

[10 Points]

Your mentor, who also throws with random accuracy, takes a turn. You then continue to throw until your score exceeds the mentor's. Let N represent the number of throws required to achieve this.

(c)

(i) Show that $P(N = 5) = \left(\frac{1}{6}\right)\left(\frac{1}{5}\right)$.

(ii) Verify that the sum of probabilities of all possible N values equals 1.

(iii) Show that $E(N) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$.

(iv) By comparing $E(N)$ with $1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \ldots$, or using an alternative approach, demonstrate that $E(N) = \infty$.

(v) Discuss whether the result $E(N) = \infty$ is practically reasonable, providing justification.

[11 Points]

Your mentor challenges you to design a non-circular target so that S follows an approximately normal distribution.

(d)

(i) Is it possible to design such a target? If yes, sketch the shape; if no, explain why not.

[3 Points]

Practice Problem - 2

Problem 2

Maximum Points: 31

This question explores the Maclaurin series expansion of e^x and its applications.

Part (a)

(i) Express the Maclaurin series expansion of e^x using sigma notation.

[1 Point]

Part (b)

(i) Use the Maclaurin series to show that $e > 2.5$.

[2 Points]

Part (c)

(i) Prove by induction that $2^{n-1} < n!$ for $n \in \mathbb{Z}$, $n \geq 3$.

(ii) By comparing with a suitable geometric series, or another method, show that $e < 3$.

[9 Points]

Part (d)

The probability density function of the normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Given that $Z \sim N(0,1)$, show that for small a,

$$P(0 < Z < a) \approx \frac{a}{\sqrt{2\pi}}.$$

[4 Points]

Part (e)

(i) Show that $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} > 0$ for all $x \in \mathbb{R}$.

[1 Point]

Part (f)

(i) Determine, with justification, whether $g(x) = \sum_{k=0}^{n} \frac{x^k}{k!} > 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

[2 Points]

Part (g)

Define the function $h(x)$ as:

$$h(x) = \sum_{k=0}^{2n} \frac{x^k}{k!}, \quad n \in \mathbb{Z}^+.$$

(i) By considering $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$, or another approach, show that $h(x)$ has a minimum point.

(ii) Show that at this minimum point, $h(x) = \frac{x^{2n}}{(2n)!}$.

[6 Points]

Part (h)

(i) Show that $h(x) > 0$ for all $x \in \mathbb{R}$.

[6 Points]

Practice Problem - 3

Problem 3

Maximum Points: 27
This question explores the fundamental properties and derivatives of hyperbolic functions $\cosh x$ and $\sinh x$. Consider a hyperbola defined by the equation $x^2 - y^2 = 1$. Draw line segments from $(0, 0)$ to (p, q) and $(p, -q)$ on the right side of the hyperbola. Define the enclosed area as A if $q \geq 0$, or $-A$ if $q < 0$. Define the hyperbolic functions as:

$$\cosh A = p,$$
$$\sinh A = q.$$

Part (a)

(i) Show that the enclosed area A is given by:

$$A = 2\int_0^q \left(\sqrt{y^2+1} - \frac{\sqrt{q^2+1}}{q} y \right) dy.$$

[3 Points]

Part (b)

(i) Compute the derivative of the function:

$$f(y) = y\sqrt{y^2+1} + \ln(y + \sqrt{y^2+1})$$

and simplify the expression.

(ii) Using the above result, demonstrate that:

$$A = \ln(q + \sqrt{q^2+1}).$$

[7 Points]

Part (c)

(i) Show that the hyperbolic sine and cosine functions satisfy the identities:

$$\sinh A = \frac{e^A - e^{-A}}{2},$$
$$\cosh A = \frac{e^A + e^{-A}}{2}.$$

[6 Points]

Part (d)

(i) Prove the fundamental identity of hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1.$$

(ii) Compute the derivatives:

$$\frac{d}{dx}\cosh x = \sinh x,$$
$$\frac{d}{dx}\sinh x = \cosh x.$$

(iii) Given the definition of the hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x},$$

find its derivative and simplify the result.

[7 Points]

Part (e)

(i) Show that the integral:

$$\int \frac{1}{\sqrt{1+x^2}}dx$$

evaluates to:

$$\sinh^{-1} x + C.$$

[4 Points]

Check Your Understanding!

Problem 1

Problem Statement

1. True/False: If a random variable X has zero variance, then X must be a constant.

2. Determine whether the following statement is **True** or **False**:

$$\text{Var}(X) = \sum x^2 P(X = x)$$

Problem 2

Problem Statement

A researcher wants to conduct a **stratified sample** of students from a university with the following population distribution across three faculties:

- **Science**: 1200 students
- **Arts**: 800 students
- **Commerce**: 1000 students

The researcher decides to take a stratified random sample of **150 students**.

Tasks

1. Calculate the number of students to be sampled from each faculty **proportionally**.
2. If the researcher mistakenly sampled **50 students from each faculty**, explain why this could introduce bias.
3. Suppose the researcher increases the total sample size to **200 students** while maintaining proportionality. Calculate the new sample sizes for each faculty.

Formula to Use

Let N_{total} be the total population:

$$N_{\text{total}} = 1200 + 800 + 1000 = 3000$$

The proportion of students sampled from each faculty should be:

$$\text{Sample Size for a Faculty} = \left(\frac{\text{Faculty Population}}{N_{\text{total}}}\right) \times \text{Total Sample Size}$$

Problem 3

Problem Statement

A teacher records the test scores of **40 students** and summarizes the data into the following frequency table:

Score (x_i)	Frequency (f_i)
10 – 20	5
20 – 30	8
30 – 40	12
40 – 50	9
50 – 60	6

Using this data, solve the following questions:

Part A: Mean and Estimation

1. **Estimate the mean test score** using the midpoint method for grouped data:
$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} \tag{1}$$

2. The teacher wants to **curve the scores** by adding 5 points to each student's score. Find the **new mean** after this transformation.

Part B: Median and Mode

1. **Estimate the median** using cumulative frequency and interpolation:

$$\text{Median} = L + \left(\frac{\frac{N}{2} - F}{f} \right) h \tag{2}$$

where:

- L = lower boundary of the median class
- N = total number of students
- F = cumulative frequency before the median class
- f = frequency of the median class
- h = class width

2. **Determine the modal class** and explain why it represents the mode.

Part C: Measures of Dispersion

1. **Estimate the standard deviation** using the formula:

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} \tag{3}$$

and interpret its meaning.

2. Find the **interquartile range (IQR)** using the cumulative frequency distribution and explain what it tells about the data spread.

3. If each student's score was **multiplied by 2**, how would this affect:

- The **mean**
- The **variance**
- The **standard deviation**

Bonus: Difficult Question

1. If the **highest 10% of scores** are considered "excellent", estimate the minimum score required for a student to be in this category.

Problem 4

Problem Statement

A researcher collects data on **advertising expenses** (in thousand dollars) and **monthly revenue** (in million dollars) for 10 different businesses. The data is summarized as follows:

$$\sum x = 250, \quad \sum y = 32, \quad \sum xy = 910, \quad \sum x^2 = 6800, \quad \sum y^2 = 110$$

where x represents the **advertising expense** and y represents the **monthly revenue**.

(a) Compute the **Pearson correlation coefficient** r. Interpret its value in terms of the strength and direction of the relationship between advertising expenses and monthly revenue.

(b) Determine the **least squares regression equation** $y = a + bx$ for predicting monthly revenue based on advertising expenses.

(c) Assess whether the regression model can be used to predict revenue for a company that spends **$400,000** on advertising. Justify your response using the concept of **extrapolation**.

(d) If a business spends **$100,000** on advertising, use the regression model to estimate its monthly revenue.

(e) A business owner claims, *"Increasing our advertising budget will always increase our revenue."* Critically evaluate this claim using the principle of **correlation vs. causation**.

Problem 5

Problem Statement

A researcher collects the following data on the number of hours 15 students spend studying per week:

$$3, 7, 12, 5, 9, 15, 20, 6, 10, 8, 11, 13, 16, 18, 14$$

(a) Arrange the data in **ascending order**.

(b) Determine the **minimum** and **maximum** values.

(c) Find the **median**, **first quartile** (Q1), and **third quartile** (Q3).

(d) Compute the **interquartile range** (IQR) and identify any potential outliers using the formula:

$$\text{Lower Bound} = Q1 - 1.5 \times IQR, \quad \text{Upper Bound} = Q3 + 1.5 \times IQR$$

(e) Draw a **box and whisker plot** for the given data.

Problem 6

Problem Statement

Problem: A continuous random variable X has the probability density function (pdf):

$$f(x) = \begin{cases} kx(2-x), & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k that makes $f(x)$ a valid probability density function.

(b) Compute the expected value $E(X)$.

(c) Find the variance $\text{Var}(X)$ using the formula:

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

(d) Determine the median m, where:

$$\int_0^m f(x)dx = \frac{1}{2}.$$

(e) If $Y = 3X + 5$, find $E(Y)$ and $\text{Var}(Y)$.

Problem 7

Problem Statement

Problem: A medical test is used to detect a rare disease. The probability that a randomly chosen person has the disease is 0.01. The test is not perfect and gives the following probabilities:

- If a person has the disease, the test correctly detects it with probability 0.95.
- If a person does not have the disease, the test incorrectly detects the disease with probability 0.05.

Define:

- D as the event that a person has the disease.
- $\bar{D}$ as the event that a person does not have the disease.
- T as the event that the test is positive.

(a) Find $P(T)$, the probability that the test is positive.

(b) Use Bayes' Theorem to find $P(D|T)$, the probability that a person actually has the disease given that they tested positive.

(c) Interpret the result in the context of medical testing. How reliable is the test in diagnosing the disease?

Problem 8

Problem Statement

Problem: A researcher studies the relationship between students' study time (Y, in hours per week) and their test scores (X, in percentage). The following summary statistics are obtained from a sample of 10 students:

$$\sum X = 750, \quad \sum Y = 120, \quad \sum XY = 9100, \quad \sum Y^2 = 1600, \quad \sum X^2 = 57500.$$

The regression line of x on y is given by:

$$x = a + by$$

(a) Find the values of a and b using the least squares method.

(b) Using your regression equation, predict the test score (x) of a student who studies for 15 hours per week.

(c) Discuss the limitations of using this regression model for predictions beyond the observed data range.

Problem 9

Problem Statement

Problem:
The heights of adult males in a certain country are normally distributed with a mean of $\mu = 175$ cm and a standard deviation of $\sigma = 8$ cm.

(a) Find the probability that a randomly selected adult male has a height between 167 cm and 183 cm.

(b) A professional basketball team only recruits players in the top 5% of the height distribution. Find the minimum height required for a player to qualify.

(c) If the heights of adult females in the same country follow a normal distribution with mean 162 cm and standard deviation 7 cm, determine the probability that a randomly chosen female is taller than a randomly chosen male.

(d) Explain why using the normal distribution might be an appropriate or inappropriate model for this scenario.

Problem 10

Problem Statement

Problem:
A manufacturing company produces electronic chips, and each chip has a probability of $p = 0.02$ of being defective. A batch contains $n = 50$ chips.

(a) Define the random variable X and justify why X follows a binomial distribution.

(b) Find the probability that exactly 3 chips in the batch are defective.

(c) Find the probability that at most 2 chips are defective.

(d) Calculate the expected number of defective chips and its standard deviation.

(e) The batch is accepted if no more than 1 chip is defective. What is the probability that the batch is rejected?

(f) If the batch size increases to $n = 500$, approximate the probability of having more than 15 defective chips using a normal approximation.

Problem 11

Problem Statement

Problem:
A fair game is played where a player rolls a biased six-sided die with faces labeled $1, 2, 3, 4, 5, 6$. The probabilities of rolling each face are given by:

$$P(X = x) = \begin{cases} \frac{1}{8}, & x = 1 \\ \frac{1}{4}, & x = 2 \\ \frac{1}{4}, & x = 3 \\ \frac{1}{8}, & x = 4 \\ \frac{1}{6}, & x = 5 \\ \frac{1}{12}, & x = 6 \end{cases}$$

The player wins or loses an amount of money based on the outcome:

- If $X = 1$, the player loses 10.
- If $X = 2$ or $X = 3$, the player wins 5.
- If $X = 4$, the player wins 10.
- If $X = 5$, the player wins 20.
- If $X = 6$, the player wins 50.

Let Y be the amount won or lost in a single game.

(a) Construct the probability distribution of Y.

(b) Verify that the total probability sums to 1.

(c) Compute $E(Y)$ and interpret its meaning in the context of the game.

(d) Determine whether the game is fair.

(e) Suppose the game is not fair. What should the organizer change in the payout structure to make it fair?

Problem 12

Problem Statement

Problem:
A company is hiring employees based on a two-step selection process: a written test and an interview. The probabilities for different outcomes are given below:

- The probability that a candidate passes the written test is $P(W) = 0.6$.
- If a candidate passes the written test, the probability of passing the interview is $P(I|W) = 0.7$.
- If a candidate fails the written test, the probability of still passing the interview is $P(I|W^c) = 0.2$.

(a) Draw a tree diagram to represent this scenario.

(b) Calculate the probability that a randomly chosen candidate passes both the written test and the interview.

(c) Find the probability that a candidate is selected (i.e., passes the interview).

(d) Given that a candidate was selected, what is the probability that they passed the written test?

(e) Are passing the written test and passing the interview independent events? Justify mathematically.

Problem 13

Matching Problem: Sampling Techniques and Their Characteristics

Match each sampling technique in **Column I** with its corresponding characteristic in **Column II**.

Sampling Technique	Characteristic
1. Simple Random Sampling	Each member has an equal chance of selection, reducing bias.
2. Convenience Sampling	Selects individuals based on availability, leading to high bias.
3. Systematic Sampling	Chooses every kth individual; efficient but sensitive to patterns.
4. Quota Sampling	Ensures a specific number from each group but lacks randomness.
5. Stratified Sampling	Divides the population into subgroups and selects proportionally.

Conclusion

Mathematics is not just about understanding theory; it's about applying concepts to solve problems effectively. This guide has provided you with a comprehensive overview of key topics, advanced problem-solving techniques, and examiner-style solutions tailored for IB Math: Analysis & Approaches HL Paper 3. However, true mastery comes from consistent practice and tackling a variety of challenging problems.

As you prepare for your exams, remember:

- **Practice is the key to success**: The more problems you solve, the more confident and efficient you become. Focus on understanding the logic behind each solution rather than memorizing formulas.
- **Learn from mistakes**: Every mistake is an opportunity to grow. Analyze where you went wrong and refine your approach.
- **Time management is crucial**: Simulate exam conditions to improve your speed and accuracy under pressure.

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Founder, Mathematics Elevate Academy
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Thank You!

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