International Baccalaureate (IB) Diploma Programme Mathematics Analysis and Approaches Higher Level

Geometry & Trigonometry

The IB 7-Scorer's Ultimate Guide

Crafted Exclusively for High-Achieving IB Mathematics Students: April 2025 Edition

Mathematics Elevate Academy

Excellence in Advanced Mathematics Education

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Disclaimer

Introduction

The IB 7-Scorer's Ultimate Guide — April 2025 Edition is thoughtfully designed for IB DP Mathematics students striving to excel in IB DP Mathematics AA SL/HL, with a special emphasis on Geometry & Trigonometry and conceptual mastery. This guide offers a wealth of expertly crafted high-level Geometry & Trigonometry problems, conceptual challenges, and much more.

Explore examiner-style solutions, detailed marking scheme breakdowns, and insightful commentary on common errors to refine your problem-solving skills. Each problem is designed to test your grasp of Geometry & Trigonometry concepts.

This guide goes beyond the IB syllabus, offering enrichment problems that challenge your mathematical thinking and prepare you for Olympiads and university-level mathematics. The solutions are presented with step-by-step clarity, expert insights, and advanced techniques, ensuring a comprehensive and engaging learning experience.

For answers or detailed solutions, keep following me — they will be available soon! For personalized learning, book a one-on-one mentorship session with me to receive customized guidance on mastering IB DP Mathematics AA/AI HL, Geometry & Trigonometry, or even Olympiad-level problems. Together, we will build the confidence and skills you need to excel.

Check Your Understanding!

1 Recall: Basic Geometry Problems Based on Properties of Angles, Circles, Triangles, and Straight Lines

Problem 1.1: Properties of Angles

Problem Statement

1. In $\triangle ABC$, $\angle A = 50°$ and $\angle B = 60°$. Find $\angle C$.

2. Two parallel lines are cut by a transversal. If one of the alternate interior angles is $65°$, find the other alternate interior angle.

3. Prove that the sum of the interior angles of a triangle is $180°$.

4. In a quadrilateral, the angles are in the ratio $2 : 3 : 4 : 5$. Find the measure of each angle.

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Problem 1.2: Properties of Circles

Problem Statement

1. Prove that the angle subtended by a diameter of a circle at any point on the circle is a right angle.

2. In a circle, two chords AB and CD intersect at P inside the circle. If $AP = 4$, $PB = 6$, and $CP = 3$, find PD.

3. A tangent to a circle is perpendicular to the radius at the point of tangency. Prove this property.

4. In a circle, the length of a chord is 8 cm, and the perpendicular distance from the center to the chord is 3 cm. Find the radius of the circle.

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Problem 1.3: Properties of Triangles

Problem Statement

1. Prove that the medians of a triangle intersect at a single point (the centroid) and that the centroid divides each median in the ratio $2:1$.

2. In $\triangle ABC$, $AB = 7$ cm, $BC = 9$ cm, and $CA = 6$ cm. Determine whether the triangle is acute, right, or obtuse.

3. Prove that the sum of any two sides of a triangle is greater than the third side.

4. In $\triangle ABC$, the angle bisector of $\angle A$ divides the opposite side BC into two segments BD and DC such that $BD : DC = AB : AC$. Prove this property.

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Problem 1.4: Properties of Straight Lines

Problem Statement

1. Find the equation of a straight line passing through the point $(2, 3)$ and parallel to the line $3x - 4y + 5 = 0$.

2. Prove that the perpendicular distance from a point (x_1, y_1) to the line $ax + by + c = 0$ is given by:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

3. Find the point of intersection of the lines $2x + 3y = 6$ and $x - y = 4$.

4. Prove that the sum of the interior angles on the same side of a transversal cutting two parallel lines is $180°$.

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Marking Guidelines

Marking Scheme

Problem 1.1: Properties of Angles

- Correct calculation of angles in triangles and quadrilaterals [2 marks per part].
- Valid proofs of angle properties [3 marks per proof].

Problem 1.2: Properties of Circles

- Correct application of circle theorems [2 marks per part].
- Valid proofs of circle properties [3 marks per proof].
- Accurate calculation of chord lengths and radii [2 marks per part].

Problem 1.3: Properties of Triangles

- Correct application of triangle properties [2 marks per part].
- Valid proofs of triangle theorems [3 marks per proof].
- Accurate classification of triangles [2 marks per part].

Problem 1.4: Properties of Straight Lines

- Correct calculation of line equations and intersections [2 marks per part].
- Valid proofs of line properties [3 marks per proof].
- Accurate application of distance formula [2 marks per part].

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Key Formulas and Theorems

Key Formulas and Theorems

1. **Sum of Angles in a Triangle**:

$$\angle A + \angle B + \angle C = 180^\circ$$

2. **Circle Theorems**:

   - Angle subtended by a diameter is 90°.
   - Tangent to a circle is perpendicular to the radius at the point of tangency.

3. **Triangle Inequality**:

$$AB + BC > AC, \quad AB + AC > BC, \quad BC + AC > AB$$

4. **Distance from a Point to a Line**:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

5. **Equation of a Line**:

$$y - y_1 = m(x - x_1)$$

where m is the slope of the line.

2 Distance and Midpoints in 3D Space

Problem 2.1: Distance Between Points

Problem Statement

1. Find the distance between the points $A(1, -2, 3)$ and $B(4, 1, -1)$. Show all your working.

2. Given that points $P(2, k, 1)$ and $Q(-1, 3, 4)$ are 5 units apart, find the value of k.

3. Determine whether the points $R(0, 2, -1)$, $S(3, -1, 2)$, and $T(6, -4, 5)$ are collinear. Justify your answer using distances.

Problem 2.2: Midpoints in 3D

Problem Statement

1. Find the midpoint of the line segment joining points $A(-2, 3, 1)$ and $B(4, -1, 5)$.
2. Point M is the midpoint of line segment PQ. If $P(3, -1, 2)$ and $M(0, 2, -1)$, find the coordinates of point Q.
3. Prove that the diagonals of a parallelogram bisect each other using 3D coordinates.

Problem 2.3: Combined Applications

Problem Statement

A tetrahedron ABCD has vertices at $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, and $D(2, 2, 2)$.

1. Find the lengths of all edges of the tetrahedron.
2. Find the coordinates of the midpoint of each edge.
3. Show that the midpoints of any two edges are equidistant from the centroid of the tetrahedron.

Formula to Use

Key Formulas

1. Distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

2. Midpoint of a line segment:

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

3. Three points are collinear if and only if:

$$d_{AB} + d_{BC} = d_{AC}$$

Marking Guidelines

Marking Scheme

Problem 2.1: Distance Between Points

- Correct substitution into distance formula [2 marks]
- Accurate calculation of squared differences [2 marks]
- Correct final distance [2 marks]
- Valid solution for k with proper working [4 marks]

Problem 2.2: Midpoints in 3D

- Correct use of midpoint formula [2 marks]
- Accurate calculation of coordinates [2 marks]
- Finding point Q using midpoint relationships [3 marks]
- Complete proof for parallelogram [3 marks]

Problem 2.3: Combined Applications

- Correct calculation of edge lengths [3 marks]
- Accurate midpoint coordinates [3 marks]
- Valid proof of equidistance [4 marks]
- Clear presentation and reasoning [2 marks]

3 Volume and Surface Area of 3D Solids

Problem 3.1: Spheres

Problem Statement

1. A sphere has a surface area of 100 square units. Find:

1. The radius of the sphere
2. The volume of the sphere
3. The ratio of its surface area to its volume

2. Two spheres have radii in the ratio 2:3. Find:

1. The ratio of their surface areas
2. The ratio of their volumes

3. A spherical balloon is being inflated so that its radius increases at a rate of 2 cm/s. At what rate is:

1. The surface area increasing when the radius is 5 cm?
2. The volume increasing when the radius is 5 cm?

Problem 3.2: Right Circular Cones

Problem Statement

1. A right circular cone has radius 6 cm and height 8 cm. Calculate:

1. The slant height of the cone
2. The curved surface area
3. The total surface area (including base)
4. The volume

2. A cone has a fixed slant height of 10 units. Find the radius and height that will:

1. Maximize the volume
2. Maximize the total surface area

3. The radius of a cone's base is increasing at 3 cm/s while its height remains constant at 12 cm. Find the rate of change of:

1. The volume when the radius is 4 cm
2. The curved surface area when the radius is 4 cm

Problem 3.3: Right Pyramids and Combined Solids

Problem Statement

1. A right square pyramid has base side length 6 cm and height 8 cm. Calculate:

1. The volume of the pyramid
2. The slant height of the pyramid
3. The total surface area

2. A solid consists of a hemisphere placed on top of a cylinder with the same radius. If the radius is 5 cm and the cylinder height is 12 cm, find:

1. The total volume of the solid
2. The total surface area of the solid

3. A cone with base radius 4 cm and height 9 cm has a sphere of radius 2 cm placed inside it such that the sphere touches the cone at its base center. Find:

1. The volume of the space between the cone and sphere
2. The height at which the sphere should be placed to maximize the remaining volume

Formula to Use

Key Formulas

1. Sphere:
   - Volume: $V = \frac{4}{3}\pi r^3$
   - Surface Area: $A = 4\pi r^2$
2. Right Circular Cone:
   - Volume: $V = \frac{1}{3}\pi r^2 h$
   - Curved Surface Area: $A = \pi r l$
   - Slant height: $l = \sqrt{r^2 + h^2}$
3. Right Pyramid:
   - Volume: $V = \frac{1}{3}Ah$
   - A is the area of the base
   - h is the height
4. Rates of Change:
   - $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$
   - $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

Marking Guidelines

Marking Scheme

Problem 3.1: Spheres

- Correct use of surface area formula [2 marks]
- Accurate calculation of radius [2 marks]
- Correct volume calculation [2 marks]
- Valid ratio calculations [2 marks]
- Accurate rates of change [2 marks]

Problem 3.2: Right Circular Cones

- Correct slant height calculation [2 marks]
- Accurate surface area calculations [3 marks]
- Correct volume calculation [2 marks]
- Valid optimization approach [3 marks]
- Accurate rates of change [2 marks]

Problem 3.3: Pyramids and Combined Solids

- Correct pyramid calculations [3 marks]
- Accurate combined solid analysis [3 marks]
- Valid volume difference calculation [3 marks]
- Correct optimization solution [3 marks]

4 Angle Between Intersecting Lines and Planes

Problem 4.1: Angle Between Two Lines in Two Dimensions

Problem Statement

1. Find the angle between the lines $y = 2x + 1$ and $y = -\frac{1}{2}x + 3$.

2. Determine whether the lines $3x - 4y + 5 = 0$ and $4x + 3y - 7 = 0$ are perpendicular. Justify your answer.

3. A line passes through the points $A(1, 2)$ and $B(4, 6)$. Find the angle between this line and the x-axis.

Problem 4.2: Angle Between a Line and a Plane

Problem Statement

1. A line has direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, and a plane has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Find the acute angle between the line and the plane.

2. A line passes through the point $P(1, 2, 3)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$. A plane is given by the equation $2x - y + z = 5$. Find the angle between the line and the plane.

3. Prove that the angle between a line and a plane is given by:

$$\sin\theta = \frac{\mathbf{d} \cdot \mathbf{n}}{\|\mathbf{d}\|\|\mathbf{n}\|}$$

where $\mathbf{d}$ is the direction vector of the line and $\mathbf{n}$ is the normal vector of the plane.

Problem 4.3: Angle Between Two Intersecting Lines in Three Dimensions

Problem Statement

1. Find the angle between the lines:

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{4} \quad \text{and} \quad \frac{x}{1} = \frac{y-2}{2} = \frac{z+1}{3}.$$

2. Two lines have direction vectors $\mathbf{d_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{d_2} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$. Find the acute angle between the lines.

3. Prove that the angle θ between two intersecting lines in three dimensions is given by:

$$\cos\theta = \frac{\mathbf{d_1} \cdot \mathbf{d_2}}{\|\mathbf{d_1}\|\|\mathbf{d_2}\|}$$

where $\mathbf{d_1}$ and $\mathbf{d_2}$ are the direction vectors of the lines.

Formula to Use

Key Formulas

1. **Angle Between Two Lines in 2D**:

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

where m_1 and m_2 are the slopes of the lines.

2. **Angle Between a Line and a Plane**:

$$\sin\theta = \frac{\mathbf{d} \cdot \mathbf{n}}{\|\mathbf{d}\|\|\mathbf{n}\|}$$

where $\mathbf{d}$ is the direction vector of the line and $\mathbf{n}$ is the normal vector of the plane.

3. **Angle Between Two Lines in 3D**:

$$\cos\theta = \frac{\mathbf{d_1} \cdot \mathbf{d_2}}{\|\mathbf{d_1}\|\|\mathbf{d_2}\|}$$

where $\mathbf{d_1}$ and $\mathbf{d_2}$ are the direction vectors of the lines.

Marking Guidelines

Marking Scheme

Problem 4.1: Angle Between Two Lines in 2D

- Correct slope calculations [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct final angle with justification [2 marks]

Problem 4.2: Angle Between a Line and a Plane

- Correct calculation of direction and normal vectors [2 marks]
- Accurate dot product and magnitudes [2 marks]
- Correct angle calculation [2 marks]
- Valid proof of the formula [2 marks]

Problem 4.3: Angle Between Two Intersecting Lines in 3D

- Correct direction vector identification [2 marks]
- Accurate dot product and magnitudes [2 marks]
- Correct final angle with justification [2 marks]
- Valid proof of the formula [2 marks]

5 Trigonometry in Right-Angled Triangles

Problem 5.1: Finding Sides in Right-Angled Triangles

Problem Statement

1. In a right-angled triangle, one of the angles is $30°$ and the hypotenuse is 10 cm. Find:

1. The length of the side opposite the $30°$ angle.
2. The length of the side adjacent to the $30°$ angle.

2. A ladder leans against a wall, making an angle of $60°$ with the ground. If the ladder is 5 m long, find:

1. The height of the wall the ladder reaches.
2. The distance of the base of the ladder from the wall.

3. A ramp is inclined at an angle of $15°$ to the horizontal. If the ramp is 8 m long, find the vertical height it covers.

Problem 5.2: Finding Angles in Right-Angled Triangles

Problem Statement

1. In a right-angled triangle, the side opposite an angle is 5 cm, and the hypotenuse is 13 cm. Find the angle.

2. A right-angled triangle has an adjacent side of 7 cm and an opposite side of 24 cm. Find the angle between the adjacent side and the hypotenuse.

3. A ramp is inclined at an angle θ to the horizontal. If the ramp is 10 m long and the vertical height is 6 m, find the angle θ.

Problem 5.3: Combined Applications

Problem Statement

1. A flagpole casts a shadow 12 m long when the angle of elevation of the sun is $45°$. Find the height of the flagpole.

2. A ship is anchored 100 m away from a lighthouse. The angle of elevation from the ship to the top of the lighthouse is $30°$. Find the height of the lighthouse.

3. A person standing on a hill observes a car at an angle of depression of $20°$. If the hill is 50 m high, find the horizontal distance of the car from the base of the hill.

Formula to Use

Key Formulas

1. **Sine, Cosine, and Tangent Ratios**:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

2. **Finding Angles**:

$$\theta = \sin^{-1}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right), \quad \theta = \cos^{-1}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right), \quad \theta = \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

Marking Guidelines

Marking Scheme

Problem 5.1: Finding Sides in Right-Angled Triangles

- Correct identification of the trigonometric ratio to use [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the side length [2 marks]

Problem 5.2: Finding Angles in Right-Angled Triangles

- Correct identification of the trigonometric ratio to use [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the angle [2 marks]

Problem 5.3: Combined Applications

- Correct setup of the problem using a diagram or trigonometric ratio [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the required length or angle [2 marks]

6 Trigonometry in Non-Right-Angled Triangles

Problem 6.1: Using the Sine Rule

Problem Statement

1. In $\triangle ABC$, $A = 40°$, $B = 60°$, and $a = 8$ cm. Find the length of side b.

2. In $\triangle PQR$, $P = 50°$, $Q = 70°$, and $q = 10$ cm. Find the length of side p.

3. In $\triangle XYZ$, $x = 7$ cm, $y = 9$ cm, and $Z = 40°$. Find angle X.

Problem 6.2: Using the Cosine Rule

Problem Statement

1. In $\triangle ABC$, $a = 5$ cm, $b = 7$ cm, and $C = 60°$. Find the length of side c.

2. In $\triangle PQR$, $p = 8$ cm, $q = 6$ cm, and $r = 10$ cm. Find angle R.

3. In $\triangle XYZ$, $x = 12$ cm, $y = 15$ cm, and $z = 18$ cm. Find the largest angle in the triangle.

Problem 6.3: Area of a Triangle Using $\frac{1}{2}ab\sin C$

Problem Statement

1. In $\triangle ABC$, $a = 6$ cm, $b = 8$ cm, and $C = 45°$. Find the area of the triangle.

2. In $\triangle PQR$, $p = 10$ cm, $q = 12$ cm, and $R = 30°$. Find the area of the triangle.

3. A triangle has sides $a = 9$ cm, $b = 12$ cm, and angle $C = 60°$. Find the area of the triangle and verify your result using the cosine rule to calculate the height.

Formula to Use

Key Formulas

1. **Sine Rule**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. **Cosine Rule**:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

3. **Area of a Triangle**:

$$\text{Area} = \frac{1}{2}ab\sin C$$

Marking Guidelines

Marking Scheme

Problem 6.1: Using the Sine Rule

- Correct identification of the sine rule [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the required length or angle [2 marks]

Problem 6.2: Using the Cosine Rule

- Correct identification of the cosine rule [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the required length or angle [2 marks]

Problem 6.3: Area of a Triangle Using $\frac{1}{2}ab\sin C$

- Correct identification of the area formula [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the area [2 marks]
- Verification using the cosine rule (if applicable) [2 marks]

7 Applications of Trigonometry

Problem 7.1: Angles of Elevation and Depression

Problem Statement

1. A person standing 50 m away from the base of a tower observes the top of the tower at an angle of elevation of $30°$. Find the height of the tower.

2. A ship is anchored 200 m away from a lighthouse. The angle of depression from the top of the lighthouse to the ship is $20°$. Find the height of the lighthouse.

3. A drone is flying at a height of 100 m above the ground. The angle of depression from the drone to a point on the ground is $45°$. Find the horizontal distance of the drone from the point on the ground.

Problem 7.2: Construction of Labelled Diagrams and Bearings

Problem Statement

1. A ship sails 10 km on a bearing of $045°$ and then 15 km on a bearing of $135°$. Find the distance of the ship from its starting point and the bearing of the ship from the starting point.

2. A plane flies 100 km on a bearing of $030°$ and then 150 km on a bearing of $120°$. Find the total distance traveled by the plane and its final position relative to the starting point.

3. Two observers are standing 500 m apart. They observe a hot air balloon at angles of elevation of $30°$ and $45°$, respectively. Find the height of the balloon and its horizontal distance from each observer.

Formula to Use

Key Formulas

1. **Basic Trigonometric Ratios**:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

2. **Bearings**: Bearings are measured clockwise from the north direction.

3. **Law of Cosines**:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

4. **Law of Sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Marking Guidelines

Marking Scheme

Problem 7.1: Angles of Elevation and Depression

- Correct identification of the trigonometric ratio to use [2 marks]
- Accurate substitution into the formula [2 marks]
- Correct calculation of the required height or distance [2 marks]

Problem 7.2: Construction of Labelled Diagrams and Bearings

- Correct construction of the labelled diagram [2 marks]
- Accurate use of trigonometric ratios or laws (sine or cosine rule) [3 marks]
- Correct calculation of the required distance, height, or bearing [3 marks]

8 Radian Measure and Applications to Circles

Problem 8.1: Converting Between Degrees and Radians

Problem Statement

1. Convert the following angles from degrees to radians:

1. $45°$
2. $120°$
3. $300°$

2. Convert the following angles from radians to degrees:

1. $\frac{\pi}{6}$
2. $\frac{2\pi}{3}$
3. 3.5 radians

3. A wheel rotates through an angle of $150°$. Express this angle in radians.

Problem 8.2: Length of an Arc

Problem Statement

1. Find the length of an arc of a circle with radius 10 cm and central angle $\frac{\pi}{3}$ radians.

2. A circle has a radius of 7 m. Find the length of an arc subtended by a central angle of $120°$.

3. A car tire has a radius of 0.4 m. If the tire rotates through an angle of 2π radians, find the distance traveled by a point on the edge of the tire.

Problem 8.3: Area of a Sector

Problem Statement

1. Find the area of a sector of a circle with radius 5 cm and central angle $\frac{\pi}{4}$ radians.

2. A circle has a radius of 12 m. Find the area of a sector subtended by a central angle of $60°$.

3. A circular pizza has a radius of 8 inches. If a slice of the pizza subtends an angle of $45°$ at the center, find the area of the slice.

Formula to Use

Key Formulas

1. **Converting Between Degrees and Radians**:

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180}, \quad \text{Degrees} = \text{Radians} \times \frac{180}{\pi}$$

2. **Length of an Arc**:

$$l = r\theta$$

where r is the radius and θ is the angle in radians.

3. **Area of a Sector**:

$$A = \frac{1}{2}r^2\theta$$

where r is the radius and θ is the angle in radians.

Marking Guidelines

Marking Scheme

Problem 8.1: Converting Between Degrees and Radians

- Correct use of the conversion formula [2 marks]
- Accurate conversion of each angle [1 mark per angle]

Problem 8.2: Length of an Arc

- Correct identification of the formula $l = r\theta$ [2 marks]
- Accurate substitution of values [2 marks]
- Correct calculation of the arc length [2 marks]

Problem 8.3: Area of a Sector

- Correct identification of the formula $A = \frac{1}{2}r^2\theta$ [2 marks]
- Accurate substitution of values [2 marks]
- Correct calculation of the area [2 marks]

9 Extending Definitions of Trigonometric Functions

Problem 9.1: Definitions of $\cos\theta$ and $\sin\theta$ Using the Unit Circle

Problem Statement

1. Using the unit circle, find the values of $\cos\theta$ and $\sin\theta$ for the following angles:

1. $\theta = 0$
2. $\theta = \frac{\pi}{2}$
3. $\theta = \pi$
4. $\theta = \frac{3\pi}{2}$

2. Verify that $\cos^2\theta + \sin^2\theta = 1$ for $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$.

3. A point P lies on the unit circle at an angle $\theta = \frac{2\pi}{3}$. Find the coordinates of P.

Problem 9.2: Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$

Problem Statement

1. Using the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$, find the value of $\tan\theta$ for:

1. $\theta = \frac{\pi}{6}$
2. $\theta = \frac{\pi}{4}$
3. $\theta = \frac{\pi}{3}$

2. Prove that $\tan\theta$ is undefined when $\cos\theta = 0$.

3. A point P lies on the unit circle at an angle $\theta = \frac{5\pi}{4}$. Find $\tan\theta$.

Problem 9.3: Exact Values of Trigonometric Ratios

Problem Statement

1. Recall the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the following angles:

$$\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}.$$

2. Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the following angles:

1. $\theta = \pi$
2. $\theta = \frac{3\pi}{2}$
3. $\theta = \frac{5\pi}{6}$

3. Prove that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ using the unit circle.

Problem 9.4: Extension of the Sine Rule to the Ambiguous Case

Problem Statement

1. In $\triangle ABC$, $a = 8$, $b = 10$, and $A = 40°$. Use the sine rule to find the two possible values of angle B.

2. In $\triangle PQR$, $p = 7$, $q = 9$, and $P = 50°$. Use the sine rule to find the two possible values of angle Q.

3. Prove that the ambiguous case arises when $a < b$ and A is acute.

Formula to Use

Key Formulas

1. **Unit Circle Definitions**:

$$\cos\theta = x, \quad \sin\theta = y$$

where (x, y) is the point on the unit circle corresponding to the angle θ.

2. **Definition of $\tan\theta$**:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{where } \cos\theta \neq 0.$$

3. **Exact Values of Trigonometric Ratios**:

$$\sin 0 = 0, \quad \sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \sin\frac{\pi}{2} = 1.$$

$$\cos 0 = 1, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \cos\frac{\pi}{2} = 0.$$

$$\tan 0 = 0, \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \quad \tan\frac{\pi}{4} = 1, \quad \tan\frac{\pi}{3} = \sqrt{3}, \quad \tan\frac{\pi}{2} \text{ is undefined.}$$

4. **Sine Rule**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Marking Guidelines

Marking Scheme

Problem 9.1: Definitions of $\cos\theta$ and $\sin\theta$ Using the Unit Circle

- Correct identification of coordinates on the unit circle [2 marks]
- Verification of $\cos^2\theta + \sin^2\theta = 1$ [2 marks]
- Accurate calculation of coordinates for given angles [2 marks]

Problem 9.2: Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$

- Correct use of the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$ [2 marks]
- Accurate calculation of $\tan\theta$ for given angles [2 marks]
- Valid proof that $\tan\theta$ is undefined when $\cos\theta = 0$ [2 marks]

Problem 9.3: Exact Values of Trigonometric Ratios

- Correct recall of exact values for standard angles [2 marks]
- Accurate calculation of exact values for additional angles [2 marks]
- Valid proof of symmetry properties using the unit circle [2 marks]

Problem 9.4: Extension of the Sine Rule to the Ambiguous Case

- Correct use of the sine rule [2 marks]
- Accurate calculation of two possible angles [2 marks]
- Valid proof of the ambiguous case conditions [2 marks]

10 Trigonometric Identities

Problem 10.1: The Pythagorean Identity

Problem Statement

1. Verify the Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$ for the following angles:

1. $\theta = 30°$
2. $\theta = 45°$
3. $\theta = 60°$

2. Simplify the following expressions using the Pythagorean identity:

1. $1 - \sin^2\theta$
2. $1 - \cos^2\theta$
3. $\cos^2\theta - \sin^2\theta$

3. Solve the equation $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$.

Problem 10.2: Double Angle Identity for Sine

Problem Statement

1. Prove the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$.

2. Find the value of $\sin 2\theta$ if:

1. $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$
2. $\sin\theta = \frac{5}{13}$ and $\cos\theta = \frac{12}{13}$

3. Solve the equation $\sin 2\theta = \frac{\sqrt{3}}{2}$ for $\theta \in [0, 2\pi]$.

Problem 10.3: Double Angle Identities for Cosine

Problem Statement

1. Prove the following forms of the double angle identity for cosine:

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta.$$

2. Find the value of $\cos 2\theta$ if:

1. $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$
2. $\sin\theta = \frac{5}{13}$ and $\cos\theta = \frac{12}{13}$

3. Solve the equation $\cos 2\theta = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$.

Formula to Use

Key Formulas

1. **Pythagorean Identity**:

$$\cos^2\theta + \sin^2\theta \equiv 1$$

2. **Double Angle Identity for Sine**:

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

3. **Double Angle Identities for Cosine**:

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$$

Marking Guidelines

Marking Scheme

Problem 10.1: The Pythagorean Identity

- Correct verification of the identity for given angles [2 marks per angle]
- Accurate simplification of expressions using the identity [2 marks per expression]
- Correct solution of the equation $\cos^2\theta + \sin^2\theta = 1$ [3 marks]

Problem 10.2: Double Angle Identity for Sine

- Correct proof of the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$ [3 marks]
- Accurate calculation of $\sin 2\theta$ for given values of $\sin\theta$ and $\cos\theta$ [2 marks per part]
- Correct solution of the equation $\sin 2\theta = \frac{\sqrt{3}}{2}$ [3 marks]

Problem 10.3: Double Angle Identities for Cosine

- Correct proof of all forms of the double angle identity for cosine [3 marks]
- Accurate calculation of $\cos 2\theta$ for given values of $\sin\theta$ and $\cos\theta$ [2 marks per part]
- Correct solution of the equation $\cos 2\theta = -\frac{1}{2}$ [3 marks]

11 Graphs of Trigonometric Functions

Problem 11.1: Basic Trigonometric Graphs

Problem Statement

1. Sketch the following graphs over the interval $[0, 2\pi]$, showing all key points:

1. $y = \sin x$
2. $y = \cos x$
3. $y = \tan x$

For each graph, state:

- The domain and range
- The period
- All x-intercepts
- All maximum and minimum points
- Any asymptotes (if applicable)

2. Explain why $\cos x$ is a horizontal translation of $\sin x$ by $\frac{\pi}{2}$ units to the left.

Problem 11.2: Transformations of Trigonometric Functions

Problem Statement

1. Sketch the following transformed graphs, showing how they relate to the basic function:

1. $y = 2\sin x$
2. $y = \cos(x - \frac{\pi}{2})$
3. $y = \tan(2x)$
4. $y = -\sin x$

2. For each transformation, state:

- The type of transformation (stretch, translation, reflection)
- The effect on the period
- The effect on the amplitude (if applicable)

Problem 11.3: Composite Trigonometric Functions

Problem Statement

1. Sketch the graphs of:

1. $y = 3\sin(2x - \pi) + 1$
2. $y = 2\cos(\frac{x}{2} + \frac{\pi}{3}) - 1$

For each graph, state:

- The amplitude
- The period
- The phase shift
- The vertical shift

2. Given the graph of $f(x) = a\sin(b(x + c)) + d$, find the values of a, b, c, and d if:

- The amplitude is 3
- The period is π
- The phase shift is $\frac{\pi}{4}$ units left
- The graph is shifted up 2 units

Problem 11.4: Real-Life Applications

Problem Statement

1. The temperature T (in °C) throughout a day can be modeled by a sinusoidal function. If the maximum temperature is 25°C at 2 PM and the minimum temperature is 15°C at 2 AM:

1. Write a trigonometric function to model the temperature.
2. Find the temperature at 6 PM.
3. When will the temperature be 20°C?

2. The height h (in meters) of a Ferris wheel above the ground can be modeled by a sinusoidal function. If the wheel has a diameter of 30 meters and makes one complete rotation every 60 seconds:

1. Write a trigonometric function to model the height.
2. Find the height after 15 seconds.
3. When will a passenger be at the highest point?

Formula to Use

Key Formulas

1. **Basic Functions**:
   - $y = \sin x$: Period 2π, Range $[-1, 1]$
   - $y = \cos x$: Period 2π, Range $[-1, 1]$
   - $y = \tan x$: Period π, Range $(-\infty, \infty)$
2. **Transformations**:
   - Amplitude: a in $y = a\sin x$ or $y = a\cos x$
   - Period: $\frac{2\pi}{|b|}$ in $y = \sin(bx)$ or $y = \cos(bx)$
   - Phase shift: $-c$ units in $y = \sin(x + c)$ or $y = \cos(x + c)$
   - Vertical shift: d in $y = \sin x + d$ or $y = \cos x + d$
3. **Composite Function**:

$$f(x) = a\sin(b(x + c)) + d \text{ or } f(x) = a\cos(b(x + c)) + d$$

where:

- a is the amplitude
- $\frac{2\pi}{|b|}$ is the period
- $-c$ is the phase shift
- d is the vertical shift

Marking Guidelines

Marking Scheme

Problem 11.1: Basic Trigonometric Graphs

- Accurate graph sketches with correct shape [2 marks each]
- Correct identification of key features [2 marks per function]
- Valid explanation of relationship between sine and cosine [2 marks]

Problem 11.2: Transformations

- Correct transformed graph sketches [2 marks each]
- Accurate description of transformations [2 marks per transformation]
- Correct analysis of effects on period and amplitude [2 marks per function]

Problem 11.3: Composite Functions

- Accurate graph sketches [3 marks each]
- Correct identification of all parameters [2 marks per function]
- Valid determination of function parameters [3 marks]

Problem 11.4: Real-Life Applications

- Correct trigonometric model [3 marks]
- Accurate calculations using the model [2 marks per calculation]
- Valid interpretation of results [2 marks]

12 Solving Trigonometric Equations

Problem 12.1: Solving Trigonometric Equations Graphically

Problem Statement

1. Use your GDC (Graphical Display Calculator) to solve the following equations in the interval $[0, 2\pi]$:

1. $\sin x = 0.5$
2. $\cos x = -0.3$
3. $\tan x = 1.2$

2. Sketch the graphs of $y = \sin x$ and $y = 0.5$ on the same axes. Use the graph to find the solutions to $\sin x = 0.5$ in $[0, 2\pi]$.

3. Explain how the intersections of the graphs of $y = \cos x$ and $y = -0.3$ correspond to the solutions of $\cos x = -0.3$.

Problem 12.2: Solving Basic Trigonometric Equations Analytically

Problem Statement

1. Solve the following equations analytically in the interval $[0, 2\pi]$:

1. $\sin x = \frac{\sqrt{3}}{2}$
2. $\cos x = -\frac{1}{2}$
3. $\tan x = 1$

2. Solve the following equations analytically in the interval $[0, 360°]$:

1. $\sin x = -0.5$
2. $\cos x = 0.8$
3. $\tan x = -1$

3. Explain why $\sin x = k$ has no solutions if $|k| > 1$.

Problem 12.3: Solving Trigonometric Equations of the Form $A = f(\theta)$

Problem Statement

1. Solve the following equations analytically in the interval $[0, 2\pi]$:

1. $\sin(2x) = 0.5$
2. $\cos(3x) = -1$
3. $\tan\left(\frac{x}{2}\right) = 1$

2. Solve the equation $\sin(2x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ in the interval $[0, 2\pi]$.

3. Solve the equation $\cos(3x - \frac{\pi}{6}) = 0$ in the interval $[0, 2\pi]$.

Problem 12.4: Using Identities to Solve Trigonometric Equations

Problem Statement

1. Solve the following equations using trigonometric identities in the interval $[0, 2\pi]$:

1. $\sin^2 x + \cos^2 x = 1$
2. $2\sin^2 x - 1 = 0$
3. $\cos 2x = 2\cos^2 x - 1$

2. Solve the equation $\sin(2x) = 2\sin x \cos x$ in the interval $[0, 2\pi]$.

3. Prove that the solutions to $\cos(2x) = 1 - 2\sin^2 x$ are the same as the solutions to $\cos(2x) = \cos^2 x - \sin^2 x$.

Problem 12.5: Solving Quadratic Trigonometric Equations

Problem Statement

1. Solve the following quadratic trigonometric equations in the interval $[0, 2\pi]$:

1. $2\sin^2 x - \sin x - 1 = 0$
2. $3\cos^2 x - 5\cos x + 2 = 0$
3. $\tan^2 x - 3\tan x + 2 = 0$

2. Solve the equation $4\sin^2 x - 4\sin x + 1 = 0$ in the interval $[0, 2\pi]$.

3. Solve the equation $2\cos^2(2x) - 3\cos(2x) + 1 = 0$ in the interval $[0, 2\pi]$.

Formula to Use

Key Formulas and Identities

1. **Basic Trigonometric Equations**:

$$\sin x = k, \quad \cos x = k, \quad \tan x = k$$

where $k \in [-1, 1]$ for $\sin x$ and $\cos x$.

2. **Trigonometric Identities**:

$$\sin^2 x + \cos^2 x = 1, \quad \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x \cos x$$

3. **Quadratic Trigonometric Equations**: Substitute $\sin x$, $\cos x$, or $\tan x$ with a variable (e.g., u) to solve as a quadratic equation.

Marking Guidelines

Marking Scheme

Problem 12.1: Solving Graphically

- Correct use of GDC to find solutions [2 marks per equation]
- Accurate graph sketches with intersections [2 marks]
- Valid explanation of graphical solutions [2 marks]

Problem 12.2: Solving Analytically

- Correct use of inverse trigonometric functions [2 marks per equation]
- Accurate calculation of all solutions in the given interval [2 marks per equation]
- Valid explanation of why $|k| > 1$ has no solutions [2 marks]

Problem 12.3: Solving $A = f(\theta)$

- Correct substitution and simplification [2 marks per equation]
- Accurate calculation of all solutions in the given interval [2 marks per equation]

Problem 12.4: Using Identities

- Correct use of trigonometric identities [2 marks per equation]
- Accurate simplification and solution [2 marks per equation]
- Valid proof of equivalence of identities [2 marks]

Problem 12.5: Solving Quadratic Trigonometric Equations

- Correct substitution to form a quadratic equation [2 marks per equation]
- Accurate solution of the quadratic equation [2 marks per equation]
- Correct back-substitution and solution of trigonometric equations [2 marks per equation]

13 Reciprocal and Inverse Trigonometric Functions

Problem 13.1: Definitions of Reciprocal Trigonometric Ratios

Problem Statement

1. Use the definitions of the reciprocal trigonometric ratios to find the following:

1. $\sec\theta$ if $\cos\theta = \frac{3}{5}$
2. $\csc\theta$ if $\sin\theta = \frac{4}{5}$
3. $\cot\theta$ if $\tan\theta = 2$

2. Prove that $\sec^2\theta - \tan^2\theta = 1$ using the definitions of $\sec\theta$ and $\tan\theta$.

3. Simplify the following expressions:

1. $\sec\theta \cdot \cos\theta$
2. $\csc^2\theta - \cot^2\theta$
3. $\cot\theta \cdot \tan\theta$

Problem 13.2: Sketching the Graphs of Reciprocal Trigonometric Functions

Problem Statement

1. Sketch the graphs of the following functions over the interval $[0, 2\pi]$, showing all key points, asymptotes, and periodicity:

1. $y = \sec x$
2. $y = \csc x$
3. $y = \cot x$

2. State the domain, range, and period of each function.

3. Explain why $y = \sec x$ and $y = \csc x$ have vertical asymptotes at certain points.

Problem 13.3: Pythagorean Identities

Problem Statement

1. Prove the following identities:

1. $1 + \tan^2\theta \equiv \sec^2\theta$
2. $1 + \cot^2\theta \equiv \csc^2\theta$

2. Solve the following equations using the Pythagorean identities:

1. $\sec^2 x - \tan^2 x = 1$
2. $\csc^2 x - \cot^2 x = 1$
3. $1 + \tan^2 x = 4$

3. Simplify the following expressions:

1. $\sec^2 x - 1$
2. $\csc^2 x - 1$
3. $\frac{\sec^2 x - 1}{\tan^2 x}$

Problem 13.4: Definitions of Inverse Trigonometric Functions

Problem Statement

1. Use the definitions of inverse trigonometric functions to find the following:

1. $\arcsin\left(\frac{1}{2}\right)$
2. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arctan(1)$

2. Solve the following equations:

1. $\arcsin x = \frac{\pi}{6}$
2. $\arccos x = \frac{\pi}{3}$
3. $\arctan x = \frac{\pi}{4}$

3. Explain the domains and ranges of $\arcsin x$, $\arccos x$, and $\arctan x$.

Problem 13.5: Sketching the Graphs of Inverse Trigonometric Functions

Problem Statement

1. Sketch the graphs of the following functions, showing all key points and asymptotes:

1. $y = \arcsin x$
2. $y = \arccos x$
3. $y = \arctan x$

2. State the domain, range, and key features of each graph.

3. Explain why the graphs of $y = \arcsin x$ and $y = \arccos x$ are reflections of each other about the line $y = \frac{\pi}{2}$.

Formula to Use

Key Formulas and Identities

1. **Reciprocal Trigonometric Ratios**:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

2. **Pythagorean Identities**:

$$1 + \tan^2\theta \equiv \sec^2\theta, \quad 1 + \cot^2\theta \equiv \csc^2\theta$$

3. **Inverse Trigonometric Functions**:

$$y = \arcsin x \implies \sin y = x, \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \arccos x \implies \cos y = x, \quad y \in [0, \pi]$$

$$y = \arctan x \implies \tan y = x, \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Marking Guidelines

Marking Scheme

Problem 13.1: Definitions of Reciprocal Trigonometric Ratios

- Correct use of definitions [2 marks per part]
- Accurate simplifications and proofs [2 marks per part]

Problem 13.2: Sketching the Graphs of Reciprocal Trigonometric Functions

- Accurate graph sketches with key points and asymptotes [3 marks per graph]
- Correct domain, range, and period [2 marks per function]

Problem 13.3: Pythagorean Identities

- Correct proofs of identities [3 marks per identity]
- Accurate solutions to equations [2 marks per equation]
- Valid simplifications [2 marks per expression]

Problem 13.4: Definitions of Inverse Trigonometric Functions

- Correct use of definitions [2 marks per part]
- Accurate solutions to equations [2 marks per equation]
- Valid explanation of domains and ranges [3 marks]

Problem 13.5: Sketching the Graphs of Inverse Trigonometric Functions

- Accurate graph sketches with key points [3 marks per graph]
- Correct domain, range, and key features [2 marks per function]

14 Compound Angle Identities

Problem 14.1: Using Compound Angle Identities

Problem Statement

1. Simplify the following expressions using compound angle identities:

1. $\sin(45° + 30°)$
2. $\cos(60° - 45°)$
3. $\tan(30° + 45°)$

2. Prove the following identities:

1. $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$
2. $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
3. $\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$

3. Solve the equation $\sin(x + \frac{\pi}{6}) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

Problem 14.2: Double Angle Identity for $\tan$

Problem Statement

1. Simplify the following expressions using the double angle identity for $\tan$:

1. $\tan(2x)$ if $\tan x = \frac{1}{2}$
2. $\tan(2x)$ if $\tan x = -\frac{\sqrt{3}}{3}$

2. Prove the identity $\tan(2\theta) \equiv \frac{2 \tan \theta}{1 - \tan^2 \theta}$.

3. Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in [0, 2\pi]$.

Problem 14.3: Applications of Compound Angle Identities

Problem Statement

1. Find the exact value of $\sin(75°)$ using the identity $\sin(A + B)$.

2. Prove that $\cos(2A) = 1 - 2\sin^2 A$ using the identity $\cos(A + B)$.

3. Solve the equation $\cos(2x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ for $x \in [0, 2\pi]$.

Formula to Use

Key Formulas and Identities

1. **Compound Angle Identities**:

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

2. **Double Angle Identity for $\tan$**:

$$\tan(2\theta) \equiv \frac{2\tan\theta}{1 - \tan^2\theta}$$

Marking Guidelines

Marking Scheme

Problem 14.1: Using Compound Angle Identities

- Correct use of compound angle identities [2 marks per part]
- Accurate simplifications and proofs [2 marks per part]
- Correct solutions to equations [3 marks]

Problem 14.2: Double Angle Identity for $\tan$

- Correct use of the double angle identity [2 marks per part]
- Accurate simplifications and proofs [2 marks per part]
- Correct solutions to equations [3 marks]

Problem 14.3: Applications of Compound Angle Identities

- Correct application of compound angle identities [2 marks per part]
- Accurate solutions to equations [3 marks]
- Valid proofs of derived identities [3 marks]

15 Symmetries of Trigonometric Graphs

Problem 15.1: Symmetry Properties of Trigonometric Graphs

Problem Statement

1. Prove the following symmetry properties of trigonometric functions using compound angle identities:

1. $\sin(-x) = -\sin x$ (odd symmetry of $\sin x$)
2. $\cos(-x) = \cos x$ (even symmetry of $\cos x$)
3. $\tan(-x) = -\tan x$ (odd symmetry of $\tan x$)

2. Verify the symmetry properties of the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ by sketching their graphs over the interval $[-2\pi, 2\pi]$.

3. Explain why the graphs of $y = \sin x$ and $y = \tan x$ are symmetric about the origin, while the graph of $y = \cos x$ is symmetric about the y-axis.

Problem 15.2: Using Compound Angle Identities to Establish Symmetry

Problem Statement

1. Use the compound angle identities to prove the following:

1. $\sin(\pi - x) = \sin x$
2. $\cos(\pi - x) = -\cos x$
3. $\tan(\pi - x) = -\tan x$

2. Prove that $\sin(\pi + x) = -\sin x$ and $\cos(\pi + x) = -\cos x$ using compound angle identities.

3. Show that $\tan(\pi + x) = \tan x$ and explain how this relates to the periodicity of the tangent function.

Problem 15.3: Applications of Symmetry Properties

Problem Statement

1. Simplify the following expressions using symmetry properties:

1. $\sin(-\frac{\pi}{3})$
2. $\cos(-\frac{\pi}{4})$
3. $\tan(-\frac{\pi}{6})$

2. Solve the equation $\sin(\pi - x) = 0.5$ for $x \in [0, 2\pi]$.

3. Prove that the graph of $y = \sin x$ is periodic with period 2π and that the graph of $y = \tan x$ is periodic with period π.

Formula to Use

Key Formulas and Identities

1. **Symmetry Properties**:

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$$

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x, \quad \tan(\pi - x) = -\tan x$$
$$\sin(\pi + x) = -\sin x, \quad \cos(\pi + x) = -\cos x, \quad \tan(\pi + x) = \tan x$$

2. **Compound Angle Identities**:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Marking Guidelines

Marking Scheme

Problem 15.1: Symmetry Properties of Trigonometric Graphs

- Correct proofs of symmetry properties using compound angle identities [3 marks per part]
- Accurate graph sketches with key points and symmetry [3 marks per graph]
- Valid explanation of symmetry properties [2 marks]

Problem 15.2: Using Compound Angle Identities to Establish Symmetry

- Correct use of compound angle identities [3 marks per part]
- Accurate proofs of symmetry properties [3 marks per part]
- Valid explanation of periodicity [2 marks]

Problem 15.3: Applications of Symmetry Properties

- Correct simplifications using symmetry properties [2 marks per part]
- Accurate solutions to equations [3 marks]
- Valid proofs of periodicity [3 marks]

16 Introduction to Vectors

Problem 16.1: Representation of Vectors as Directed Line Segments

Problem Statement

1. Express the following directed line segments as column vectors:

1. A vector from $A(2, 3)$ to $B(5, 7)$.
2. A vector from $P(-1, 4)$ to $Q(3, -2)$.
3. A vector from $X(0, 0)$ to $Y(6, -3)$.

2. Verify that the vector from $A(2, 3)$ to $B(5, 7)$ is the same as the vector from $C(1, 2)$ to $D(4, 6)$.

Problem 16.2: Base Vectors i, j, and k

Problem Statement

1. Express the following column vectors in terms of the base vectors i, j, and k:

1. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
2. $\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$
3. $\begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}$

2. Write the vector $3i - 2j + 5k$ as a column vector.

3. Verify that the vector $\begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$ can be written as $6i - 4j + 2k$.

Problem 16.3: Vector Operations

Problem Statement

1. Add and subtract the following vectors:

1. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
2. $\mathbf{a} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$

2. Multiply the following vectors by scalars:

1. $2\mathbf{u}$, where $\mathbf{u} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$
2. $-3\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

3. Determine whether the following vectors are parallel:

1. $\mathbf{p} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
2. $\mathbf{m} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

Problem 16.4: Magnitude and Unit Vectors

Problem Statement

1. Calculate the magnitude of the following vectors:

1. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

2. $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

3. $\mathbf{w} = \begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}$

2. Find a unit vector in the direction of the following vectors:

1. $\mathbf{a} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

2. $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 12 \end{bmatrix}$

3. Verify that the unit vector of $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has a magnitude of 1.

Formula to Use

Key Formulas and Definitions

1. **Column Vector Representation**: A vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by:
$$\mathbf{v} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$
2. **Base Vectors**:
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 i + v_2 j + v_3 k$$
3. **Magnitude of a Vector**:
$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$
4. **Unit Vector**: A unit vector in the direction of $\mathbf{v}$ is given by:
$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$
5. **Parallel Vectors**: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if:
$$\mathbf{u} = k\mathbf{v}, \quad \text{where } k \text{ is a scalar.}$$

Marking Guidelines

Marking Scheme

Problem 16.1: Representation of Vectors as Directed Line Segments

- Correct calculation of column vectors [2 marks per part]
- Verification of equivalent vectors [2 marks]

Problem 16.2: Base Vectors i, j, and k

- Correct conversion between column vectors and base vectors [2 marks per part]
- Verification of equivalence [2 marks]

Problem 16.3: Vector Operations

- Correct addition and subtraction of vectors [2 marks per part]
- Accurate scalar multiplication [2 marks per part]
- Correct determination of parallel vectors [2 marks per part]

Problem 16.4: Magnitude and Unit Vectors

- Correct calculation of magnitude [2 marks per part]
- Accurate calculation of unit vectors [2 marks per part]
- Verification of unit vector magnitude [2 marks]

17 Geometry and Vectors

Problem 17.1: Displacement Vectors

Problem Statement

1. Find the displacement vector $\mathbf{AB}$ for the following pairs of points:

1. $A(2, 3)$ and $B(5, 7)$
2. $P(-1, 4)$ and $Q(3, -2)$
3. $X(0, 0)$ and $Y(6, -3)$

2. Verify that the displacement vector $\mathbf{AB}$ is the same as $\mathbf{CD}$ for the points $A(1, 2)$, $B(4, 6)$, $C(0, 0)$, and $D(3, 4)$.

Problem 17.2: Distance Between Two Points

Problem Statement

1. Find the distance between the following pairs of points:

1. $A(2, 3)$ and $B(5, 7)$
2. $P(-1, 4)$ and $Q(3, -2)$
3. $X(0, 0)$ and $Y(6, -3)$

2. Prove that the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

3. Find the distance between the points $A(1, 2, 3)$ and $B(4, 6, 8)$ in 3D space.

Problem 17.3: Proofs of Geometrical Properties Using Vectors

Problem Statement

1. Prove that the diagonals of a parallelogram bisect each other using vectors.

2. Prove that the medians of a triangle intersect at a single point (the centroid) using vectors.

3. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length using vectors.

4. Show that the points $A(1, 2)$, $B(3, 6)$, and $C(5, 10)$ are collinear using vectors.

Formula to Use

Key Formulas and Definitions

1. **Displacement Vector**: The displacement vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by:

$$\mathbf{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}.$$

2. **Distance Between Two Points**: The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

3. **Collinearity**: Three points A, B, and C are collinear if the vectors $\mathbf{AB}$ and $\mathbf{AC}$ are parallel, i.e., $\mathbf{AB} = k\mathbf{AC}$ for some scalar k.

Marking Guidelines

Marking Scheme

Problem 17.1: Displacement Vectors

- Correct calculation of displacement vectors [2 marks per part]
- Verification of equivalent displacement vectors [2 marks]

Problem 17.2: Distance Between Two Points

- Correct calculation of distances [2 marks per part]
- Valid proof of the distance formula [3 marks]
- Accurate calculation of 3D distance [3 marks]

Problem 17.3: Proofs of Geometrical Properties Using Vectors

- Correct use of vector properties in proofs [3 marks per proof]
- Accurate verification of collinearity [3 marks]
- Valid explanation of geometrical results [2 marks]

18 The Scalar Product

Problem 18.1: Calculating Scalar Product Using Components

Problem Statement

1. Calculate the scalar product of the following pairs of vectors:

1. $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$

2. $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

3. $\mathbf{p} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

2. Verify that $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$ for:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Problem 18.2: Scalar Product Using Magnitude and Angle

Problem Statement

1. Calculate the scalar product of the following vectors given their magnitudes and the angle between them:

1. $|\mathbf{v}| = 3$, $|\mathbf{w}| = 4$, angle between vectors $= 60°$

2. $|\mathbf{a}| = 5$, $|\mathbf{b}| = 2$, angle between vectors $= 120°$

3. $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$, angle between vectors $= 90°$

2. If $\mathbf{v} \cdot \mathbf{w} = 10$ and $|\mathbf{v}| = 5$, $|\mathbf{w}| = 4$, find the angle between the vectors.

3. Prove that $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$ using the component form of the scalar product.

Problem 18.3: Finding Angles Between Vectors

Problem Statement

1. Find the angle between the following pairs of vectors:

1. $\mathbf{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}1\\0\\1\end{bmatrix}$
2. $\mathbf{a} = \begin{bmatrix}2\\-1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}3\\4\end{bmatrix}$
3. $\mathbf{p} = \begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$

2. Find the angle between the diagonal of a cube and one of its edges.

3. Determine if the angle between two vectors is acute, right, or obtuse if their scalar product is:

1. Positive
2. Zero
3. Negative

Problem 18.4: Perpendicular and Parallel Vectors

Problem Statement

1. Determine whether the following pairs of vectors are perpendicular or parallel:

1. $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$
2. $\mathbf{a} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3. $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$

2. Find a vector perpendicular to $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3. Prove that if two vectors are perpendicular, their scalar product is zero.

Formula to Use

Key Formulas and Properties

1. **Scalar Product (Component Form)**:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

2. **Scalar Product (Magnitude Form)**:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$$

3. **Angle Between Vectors**:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

4. **Properties**:

- Perpendicular vectors: $\mathbf{v} \cdot \mathbf{w} = 0$
- Parallel vectors: $|\mathbf{v} \cdot \mathbf{w}| = |\mathbf{v}||\mathbf{w}|$

Marking Guidelines

Marking Scheme

Problem 18.1: Calculating Scalar Product Using Components

- Correct calculation of scalar products [2 marks per part]
- Valid verification of distributive property [3 marks]

Problem 18.2: Scalar Product Using Magnitude and Angle

- Correct calculation of scalar products [2 marks per part]
- Accurate calculation of angle [3 marks]
- Valid proof of equivalence [3 marks]

Problem 18.3: Finding Angles Between Vectors

- Correct calculation of angles [2 marks per part]
- Accurate solution for cube problem [3 marks]
- Valid analysis of scalar product sign [2 marks]

Problem 18.4: Perpendicular and Parallel Vectors

- Correct determination of perpendicularity/parallelism [2 marks per part]
- Valid construction of perpendicular vector [3 marks]
- Correct proof for perpendicular vectors [3 marks]

19 Equation of a Line in 3D

Problem 19.1: Vector Equation of a Line

Problem Statement

1. Find the vector equation of a line passing through the point $A(1, 2, 3)$ and parallel to the vector $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$.

2. Find the vector equation of a line passing through the points $P(2, -1, 3)$ and $Q(5, 2, 6)$.

3. Verify that the point $R(9, -4, 7)$ lies on the line with equation:

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$$

Problem 19.2: Converting Between Forms of a Line

Problem Statement

1. Convert the following vector equations into parametric form:

1. $\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2. $\mathbf{r} = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$

2. Convert the following parametric equations into vector form:

1. $x = 1 + 2t$, $y = -3 + t$, $z = 4 - t$

2. $x = 3 - 4s$, $y = 2 + 5s$, $z = -1 + 6s$

3. Convert the following vector equations into Cartesian form:

1. $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

2. $\mathbf{r} = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Problem 19.3: Angle Between Two Lines

Problem Statement

1. Find the angle between the following pairs of lines:

1. $\mathbf{r}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2. $\mathbf{r}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} -6 \\ -8 \\ -10 \end{bmatrix}$

2. Prove that the lines $\mathbf{r}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix}$ are parallel.

Problem 19.4: Applications to Kinematics

Problem Statement

1. A particle starts at the point $A(1, 2, 3)$ and moves with a constant velocity $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$. Write the vector equation of the particle's motion.

2. Find the position of the particle at $t = 5$ seconds.

3. Two particles move along the lines:

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Determine if the particles will collide.

Formula to Use

Key Formulas and Definitions

1. **Vector Equation of a Line**:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$

where $\mathbf{a}$ is a point on the line, $\mathbf{b}$ is a vector parallel to the line, and λ is a scalar.

2. **Parametric Form**:

$$x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n$$

3. **Cartesian Form**:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

4. **Angle Between Two Lines**:

$$\cos\theta = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the lines.

5. **Kinematics**: The position of a particle moving with constant velocity $\mathbf{v}$ is given by:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ is the initial position and t is time.

Marking Guidelines

Marking Scheme

Problem 19.1: Vector Equation of a Line

- Correct calculation of vector equation [3 marks per part]
- Verification of a point on the line [2 marks]

Problem 19.2: Converting Between Forms of a Line

- Correct conversion to parametric form [2 marks per part]
- Correct conversion to vector form [2 marks per part]
- Correct conversion to Cartesian form [2 marks per part]

Problem 19.3: Angle Between Two Lines

- Correct calculation of angle using scalar product [3 marks per part]
- Valid proof of parallelism [3 marks]

Problem 19.4: Applications to Kinematics

- Correct vector equation of motion [2 marks]
- Accurate calculation of position at a given time [2 marks]
- Valid determination of collision [3 marks]

20 Intersection of Lines

Problem 20.1: Parallel and Coincident Lines

Problem Statement

1. Determine whether the following pairs of lines are parallel, coincident, or neither:

1. $\mathbf{r}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda \begin{bmatrix}2\\-1\\4\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}0\\-1\\5\end{bmatrix} + \mu \begin{bmatrix}4\\-2\\8\end{bmatrix}$

2. $\mathbf{r}_1 = \begin{bmatrix}2\\0\\-1\end{bmatrix} + \lambda \begin{bmatrix}3\\4\\5\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}-1\\2\\3\end{bmatrix} + \mu \begin{bmatrix}-6\\-8\\-10\end{bmatrix}$

3. $\mathbf{r}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} + \lambda \begin{bmatrix}2\\3\\4\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}2\\2\\2\end{bmatrix} + \mu \begin{bmatrix}1\\1.5\\2\end{bmatrix}$

2. Prove that two lines are coincident if their direction vectors are parallel and one point on one line lies on the other line.

Problem 20.2: Intersecting or Skew Lines

Problem Statement

1. Determine whether the following pairs of lines intersect or are skew:

1. $\mathbf{r}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda \begin{bmatrix}2\\-1\\4\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}0\\-1\\5\end{bmatrix} + \mu \begin{bmatrix}1\\2\\-1\end{bmatrix}$

2. $\mathbf{r}_1 = \begin{bmatrix}2\\0\\-1\end{bmatrix} + \lambda \begin{bmatrix}3\\4\\5\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}-1\\2\\3\end{bmatrix} + \mu \begin{bmatrix}-6\\-8\\-10\end{bmatrix}$

3. $\mathbf{r}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} + \lambda \begin{bmatrix}2\\3\\4\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}2\\2\\2\end{bmatrix} + \mu \begin{bmatrix}1\\1\\1\end{bmatrix}$

2. Prove that two lines are skew if their direction vectors are not parallel and they do not intersect.

Problem 20.3: Points of Intersection

Problem Statement

1. Find the point of intersection of the following pairs of lines:

1. $\mathbf{r}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda \begin{bmatrix}2\\-1\\4\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}0\\-1\\5\end{bmatrix} + \mu \begin{bmatrix}1\\2\\-1\end{bmatrix}$

2. $\mathbf{r}_1 = \begin{bmatrix}2\\0\\-1\end{bmatrix} + \lambda \begin{bmatrix}3\\4\\5\end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix}-1\\2\\3\end{bmatrix} + \mu \begin{bmatrix}-6\\-8\\-10\end{bmatrix}$

2. Verify that the point of intersection lies on both lines for the first pair of lines in part 1.

3. Explain why two skew lines do not have a point of intersection.

Formula to Use

Key Formulas and Definitions

1. **Vector Equation of a Line**:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where $\mathbf{a}$ is a point on the line, $\mathbf{b}$ is a direction vector, and λ is a scalar.

2. **Parallel Lines**: Two lines are parallel if their direction vectors are scalar multiples:

$$\mathbf{b}_1 = k\mathbf{b}_2$$

3. **Coincident Lines**: Two lines are coincident if they are parallel and one point on one line lies on the other line.

4. **Intersecting Lines**: Two lines intersect if there exist values of λ and μ such that:

$$\mathbf{a}_1 + \lambda\mathbf{b}_1 = \mathbf{a}_2 + \mu\mathbf{b}_2$$

5. **Skew Lines**: Two lines are skew if they are not parallel and do not intersect.

6. **Point of Intersection**: Solve for λ and μ in the equation:

$$\mathbf{a}_1 + \lambda\mathbf{b}_1 = \mathbf{a}_2 + \mu\mathbf{b}_2$$

and substitute back to find the point of intersection.

Marking Guidelines

Marking Scheme

Problem 20.1: Parallel and Coincident Lines

- Correct determination of parallelism [2 marks per part]
- Accurate verification of coincidence [2 marks per part]
- Valid proof of conditions for coincidence [3 marks]

Problem 20.2: Intersecting or Skew Lines

- Correct determination of intersection or skewness [3 marks per part]
- Valid proof of conditions for skewness [3 marks]

Problem 20.3: Points of Intersection

- Correct calculation of λ and μ [2 marks per part]
- Accurate determination of the point of intersection [2 marks per part]
- Verification of the point of intersection [2 marks]
- Valid explanation for skew lines [2 marks]

21 The Vector Product

Problem 21.1: Calculating the Vector Product Using Components

Problem Statement

1. Calculate the vector product $\mathbf{v} \times \mathbf{w}$ for the following pairs of vectors:

1. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$
2. $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$
3. $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

2. Verify that $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

3. Prove that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ for any vectors $\mathbf{v}$ and $\mathbf{w}$.

Problem 21.2: Magnitude of the Vector Product

Problem Statement

1. Calculate the magnitude of the vector product $|\mathbf{v} \times \mathbf{w}|$ for the following pairs of vectors:

1. $|\mathbf{v}| = 3$, $|\mathbf{w}| = 4$, angle between vectors $= 90°$
2. $|\mathbf{a}| = 5$, $|\mathbf{b}| = 2$, angle between vectors $= 60°$
3. $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$, angle between vectors $= 120°$

2. If $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, find the angle between $\mathbf{v}$ and $\mathbf{w}$ given that $|\mathbf{v}| = 5$ and $|\mathbf{w}| = 4$.

3. Prove that $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \sin\theta$ using the component definition of the vector product.

Problem 21.3: Properties of the Vector Product

Problem Statement

1. Verify the following properties of the vector product:

1. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
2. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

2. Prove that the vector product is distributive over addition:

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

3. Show that the vector product of two parallel vectors is zero.

Problem 21.4: Geometric Interpretation of the Vector Product

Problem Statement

1. Calculate the area of a parallelogram with adjacent sides $\mathbf{v}$ and $\mathbf{w}$ for the following pairs of vectors:

1. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$
2. $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

2. Prove that the area of a triangle with adjacent sides $\mathbf{v}$ and $\mathbf{w}$ is given by:

$$A = \frac{1}{2}|\mathbf{v} \times \mathbf{w}|.$$

3. Find the area of a triangle with vertices $A(1, 2, 3)$, $B(4, 5, 6)$, and $C(7, 8, 9)$ using the vector product.

Formula to Use

Key Formulas and Definitions

1. **Vector Product (Component Form)**:

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

2. **Magnitude of the Vector Product**:

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

3. **Area of a Parallelogram**:

$$A = |\mathbf{v} \times \mathbf{w}|$$

4. **Area of a Triangle**:

$$A = \frac{1}{2}|\mathbf{v} \times \mathbf{w}|$$

Marking Guidelines

Marking Scheme

Problem 21.1: Calculating the Vector Product Using Components

- Correct calculation of vector product components [3 marks per part]
- Verification of vector product properties [3 marks]
- Valid proof of orthogonality of $\mathbf{v} \times \mathbf{w}$ and $\mathbf{v}$ [3 marks]

Problem 21.2: Magnitude of the Vector Product

- Correct calculation of magnitude [2 marks per part]
- Accurate calculation of angle between vectors [3 marks]
- Valid proof of magnitude formula [3 marks]

Problem 21.3: Properties of the Vector Product

- Correct verification of properties [2 marks per part]
- Valid proof of distributive property [3 marks]
- Accurate demonstration of zero vector product for parallel vectors [3 marks]

Problem 21.4: Geometric Interpretation of the Vector Product

- Correct calculation of parallelogram area [3 marks per part]
- Valid proof of triangle area formula [3 marks]
- Accurate calculation of triangle area using vertices [3 marks]

22 Equation of a Plane

Problem 22.1: Vector Equation of a Plane

Problem Statement

1. Find the vector equation of a plane passing through the point $A(1,2,3)$ and containing the vectors $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$.

2. Find the vector equation of a plane passing through the points $P(1,0,2)$, $Q(3,1,4)$, and $R(2,-1,3)$.

3. Verify that the point $S(4,3,5)$ lies on the plane with equation:

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}.$$

Problem 22.2: Scalar Product Form of a Plane

Problem Statement

1. Find the scalar product form of the equation of a plane passing through the point $A(1,2,3)$ with normal vector $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

2. Convert the following vector equations into scalar product form:

1. $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$
2. $\mathbf{r} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. Verify that the point $P(2,3,4)$ satisfies the scalar product equation of the plane:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Problem 22.3: Cartesian Equation of a Plane

Problem Statement

1. Find the Cartesian equation of a plane passing through the point $A(1, 2, 3)$ with normal vector $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

2. Convert the following scalar product equations into Cartesian form:

1. $\mathbf{r} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14$
2. $\mathbf{r} \cdot \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = -8$

3. Verify that the point $P(3, -1, 2)$ satisfies the Cartesian equation of the plane:

$$2x - y + 4z = 10.$$

Problem 22.4: Applications of Plane Equations

Problem Statement

1. Find the intersection of the plane $2x - y + 4z = 10$ with the line:

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

2. Determine whether the planes $x + 2y - z = 5$ and $2x + 4y - 2z = 10$ are parallel, coincident, or neither.

3. Find the line of intersection of the planes $x+y+z = 6$ and $2x-y+3z = 10$.

Formula to Use

Key Formulas and Definitions

1. **Vector Equation of a Plane**:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

where $\mathbf{a}$ is a point on the plane, and $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors within the plane.

2. **Scalar Product Form of a Plane**:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

where $\mathbf{n}$ is a normal vector to the plane, and $\mathbf{a}$ is a point on the plane.

3. **Cartesian Equation of a Plane**:

$$ax + by + cz = d$$

where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the normal vector, and $d = \mathbf{a} \cdot \mathbf{n}$.

4. **Intersection of a Line and a Plane**: Substitute the parametric equations of the line into the Cartesian equation of the plane and solve for the parameter.

Marking Guidelines

Marking Scheme

Problem 22.1: Vector Equation of a Plane

- Correct calculation of vector equation [3 marks per part]
- Verification of a point on the plane [2 marks]

Problem 22.2: Scalar Product Form of a Plane

- Correct calculation of scalar product form [3 marks per part]
- Accurate conversion from vector to scalar product form [2 marks per part]
- Verification of a point on the plane [2 marks]

Problem 22.3: Cartesian Equation of a Plane

- Correct calculation of Cartesian equation [3 marks per part]
- Accurate conversion from scalar product to Cartesian form [2 marks per part]
- Verification of a point on the plane [2 marks]

Problem 22.4: Applications of Plane Equations

- Correct intersection of line and plane [3 marks]
- Accurate determination of parallelism or coincidence of planes [3 marks]
- Correct calculation of line of intersection of planes [3 marks]

Conclusion

Mathematics is not just about understanding theory; it is about applying concepts to solve problems effectively. This guide has provided you with a collection of expertly crafted practice problems focused on Geometry & Trigonometry, designed to challenge your understanding and enhance your problem-solving skills.

For detailed solutions and answers, keep following me — they will be available soon! If you're looking for personalized guidance, book a one-on-one mentorship session with me to deepen your understanding of IB Mathematics AA/AI HL, Geometry & Trigonometry, or even Olympiad-level problems. Together, we can build the confidence and skills you need to excel in mathematics.

As you prepare for your exams, remember:

- **Practice is the key to success**: The more problems you solve, the more confident and efficient you become. Focus on understanding the logic behind each solution rather than memorizing formulas.
- **Learn from mistakes**: Every mistake is an opportunity to grow. Analyze where you went wrong and refine your approach.
- **Time management is crucial**: Simulate exam conditions to improve your speed and accuracy under pressure.

If you're aiming for a guaranteed improvement and want to elevate your performance to the next level, consider applying for my **exclusive personalized mentorship program**. As an alumnus of IIT Guwahati and ISI, with over 5 years of teaching experience from the school level to university students, now mentoring high-achieving IB students, I specialize in:

- **Tailored guidance**: Customized study plans and strategies based on your strengths and weaknesses.
- **Exam-focused preparation**: Insights into examiner expectations and tips to maximize your score.
- **Beyond IB HL Problem-Solving**: My mentorship is not limited to IB HL Mathematics. I will enrich your mathematical thinking to push you toward **Olympiad-level problem-solving** and help you excel in **quantitative aptitude**, preparing you for competitive exams and real-world challenges.
- **One-on-one mentorship**: Direct support to clarify doubts, build confidence, and achieve your goals.

Join the ranks of students who have transformed their performance and achieved top scores with my mentorship. Visit www.mathematicselevateacademy.com to access free resources, book a session, or apply for the program. Let's work together to make your IB Mathematics journey a success!

"Success in mathematics comes not from the number of problems you've solved, but from the confidence you've gained in solving them."

- Rishabh Kumar
Founder, Mathematics Elevate Academy
Elite Mentor for IB Mathematics
Alumnus of IIT Guwahati & Indian Statistical Institute

Thank You!

Rishabh Kumar
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www.mathematicselevateacademy.com